Solutions to H.W # 5

1. Suppose $f \in BV[a,b]$ and $[c,d] \subset [a,b]$. Let $P = \{c = t_0, t_1, \dots, t_{n-1}, t_n = d\}$ be a partition of $[c,d]$. Define $Q = \{a = x, t_0, \dots, t_n, y = b\}$. Then Q is a partition of $[a,b]$ that contains P. Then $V(f,P) \le V(f,Q) \le V_a^b f$. Hence $V_c^d f = \sup_P V(f,P) \le V_a^b f$.

2. Suppose that $f \in BV[a,b]$ and $V_{a+\epsilon}^b f \le M$ for $\epsilon > 0$. Then, if $P = \{a = t_0, t_1, \dots, t_n = b\}$ is a partition of $[a,b]$,

$$V(f,P) = \sum_{i=1}^n |f(t_i) - f(t_{i-1})| = |f(t_1) - f(a)| + \sum_{i=2}^n |f(t_i) - f(t_{i-1})|$$

$$\le |f(t_1) - f(a)| + M \le 2\|f\|_\infty + M.$$

Therefore $V_a^b f \le 2\|f\|_\infty + M < \infty$, which implies that $f \in BV[a,b]$. It is not true, however, that $V_a^b f$ is necessarily less than M. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in (0,1] \\ 3 & \text{if } x = 0 \end{cases}$$

Then $V_\epsilon^1 f = 0 \le 1$ but $V_0^1 f = 2 > 1$.

If we assume that $f(a+) = f(a)$, the assertion $V_{a+\epsilon}^b f \le M \Rightarrow V_a^b f \le M$ will be true.

3. Suppose f is a polygonal function or a polynomial function defined on the interval $[a,b]$. Let (t_{i-1}, t_i), $i \in \{1, \dots, n\}$ be the intervals where f is monotone. Then

$$V_a^b f = \sum_{i=1}^{n} V_{t_{i-1}}^{t_i} f = \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| = \sum_{i=1}^{n} \left| \int_{t_{i-1}}^{t_i} f'(t)\,dt \right|$$

$$= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |f'(t)|\,dt = \int_a^b |f'(t)|\,dt \quad \text{where}$$

$$\left| \int_{t_{i-1}}^{t_i} f'(t)\,dt \right| = \int_{t_{i-1}}^{t_i} |f'(t)|\,dt$$ because f' does not change sign in (t_{i-1}, t_i).

4. Suppose that $f_n \mapsto f$ pointwise on $[a,b]$. Assume that each f_n is increasing. Then for $x < y$, $f_n(y) - f_n(x) \geq 0$ for all n. Hence $f(y) - f(x) = \lim_{n \to \infty} \left(f_n(y) - f_n(x) \right) \geq 0$.

Assume that f_n is of bounded variation for each n. It does not follow that the pointwise limit f is of bounded variation: Define $f_n : [a,b] \mapsto \mathbb{R}$ by $f(x) = \begin{cases} 1 & \text{if } x \in \{r_1, \dots, r_n\} \\ 0 & \text{otherwise} \end{cases}$ where $\{r_n\}_{n=1}^{\infty}$ is an enumeration of the rational numbers in $[a,b]$. Then $f_n \mapsto \chi_{\mathbb{Q}}$ pointwise on $[a,b]$. Observe that $V_a^b f_n = 2n < \infty$, but $V_a^b \chi_{\mathbb{Q}} = \infty$.

5. Suppose that $f_n \mapsto f$ pointwise on $[a,b]$. Let $P = \{a = t_0, t_1, \ldots, t_{m-1}, t_m = b\}$ be a fixed partition of $[a,b]$. Then $V(f,P) = \sum_{i=1}^{m} |f(t_i) - f(t_{i-1})| = \sum_{i=1}^{m} |\lim_{n\to\infty} (f_n(t_i) - f_n(t_{i-1}))|$

$= \sum_{i=1}^{m} \lim_{n\to\infty} |f_n(t_i) - f_n(t_{i-1})| = \lim_{n\to\infty} \sum_{i=1}^{m} |f_n(t_i) - f_n(t_{i-1})|$

$= \lim_{n\to\infty} V(f_n, P)$.

Furthermore, if we assume $V_a^b f_n \le K$ for all n, then for any partition P of $[a,b]$, $V(f,P) = \lim_{n\to\infty} V(f_n, P) \le K$ (why?) Therefore $V_a^b f = \sup_P V(f,P) \le K$.

6. Suppose $f_n \mapsto f$ pointwise on $[a,b]$. Then, for any partition P of $[a,b]$, $V(f,P) = \lim_{n\to\infty} V(f_n, P) = \liminf_{n\to\infty} V(f_n, P)$ $\le \liminf_{n\to\infty} V_a^b f_n$ where $\liminf_{n\to\infty} V(f_n,P) = \limsup_{n\to\infty} V(f_n,P) =$ $= \lim_{n\to\infty} V(f_n, P)$, because $\lim_{n\to\infty} V(f_n, P)$ exists.

We have shown that $V(f,P) \le \liminf_{n\to\infty} V_a^b f_n$ for any P. Thus $V_a^b f = \sup_P V(f,P) \le \liminf_{n\to\infty} V_a^b f_n$.

7. We show that $V_a^b f = 2\sum_{n=1}^{\infty}|c_n|$. Hence $f \in BV[a,b]$ iff $\sum_{n=1}^{\infty}|c_n| < \infty$.

For $N \in \mathbb{N}$, let P_N be a partition that contains $x_1, \dots, x_N$ as well as at least one point $t \in \{x_n : n \geq 1\}$ between x_i and x_j. Then $V(f, P_N) = 2\sum_{n=1}^{N}|c_n|$.

Observe that for any refinement $Q \supset P_N$ that does not include points x_m; $m \geq N+1$, $V(f,Q) = V(f, P_N)$. Thus $V_a^b f = 2\sum_{n=1}^{\infty}|c_n|$ as desired.

8. Let $K = \sum_{n=1}^{\infty}|c_n|$ and set $g_n : [a,b] \mapsto \mathbb{R}$ by $g_n(x) = I(x - x_n)$ and $f_N = \sum_{n=1}^{N} c_n g_n$. Observe that $\|g_n\|_\infty = 1$.

Hence, by the M-test

$$\sum_{n=1}^{\infty}\|c_n g_n\|_\infty = \sum_{n=1}^{\infty}|c_n|\|g_n\|_\infty = \sum_{n=1}^{\infty}|c_n| = K$$

so $f_N \overset{\longrightarrow}{\longmapsto} f$.

Notice that each f_N is a right-continuous step function with $c_n = f_N(x_n) - f_N(x_n^-)$. Thus $V_a^b f_N = \sum_{n=1}^{N}|c_n|$. Since $\{f_N\}_{N=1}^{\infty}$ is Cauchy, $\lim_{m \to \infty} V(f_m, P) = V(f, P)$ for any partition P of $[a,b]$.

Notice that for $\epsilon > 0$, $\lim_{m \to \infty} V(f_m - f_N, P) \leq \epsilon$ for sufficiently large N. Thus $V_a^b f \leq V_a^b(f - f_N) + V_a^b f_N \leq \epsilon + \sum_{n=1}^{N}|c_n|$

This implies that $V_a^b f = \sum_{n=1}^{\infty} |c_n|$.

9. Suppose $f \in C[a,b] \cap BV[a,b]$. Since $f \in BV[a,b]$, $f =$ $= v-(v-f)$ where $v(x) = V_a^x f$ and since $f \in C[a,b]$, $v \in [a,b]$ as well. Clearly v and $v-f$ are increasing. Let $i(x) = x$. Then $f = (v+i)-(v-f+i)$ where $v+i$ and $v-f+i$ are strictly increasing continuous functions.

10. We first show that $g \in BV[a,b]$ by expressing g as the difference of two increasing functions. Since $f \in BV[a,b]$, $f = v-(v-f)$, where v and $v-f$ are increasing functions. Clearly $g(x) = f(x+) = v(x+)-(v(x+)-f(x+)). = \omega(x)-(\omega(x)-g(x))$, where $\omega(x) = v(x+)$.

If $x<y$, $\omega(y)-\omega(x) = v(y+)-v(x+) \geq v(y-\frac{y-x}{4})-v(x+\frac{y-x}{4})$

≥ 0

Similarly, $(\omega(y)-g(y))-(\omega(x)-g(x)) \geq \left(v(y-\frac{y-x}{4})-f(y-\frac{y-x}{4})\right)-$
$-\left(v(x+\frac{y-x}{4})-f(x+\frac{y-x}{4})\right) \geq 0$

Thus ω, and $\omega-g$ are increasing as desired.

Finally, $g(x+) = \lim_{\delta \to 0^+} g(x+\delta) = \lim_{\delta \to 0^+} f(x+\delta+) =$

$= \lim_{\delta \to 0^+} \lim_{\epsilon \to 0^+} f(x+\delta+\epsilon) = \lim_{k \to 0^+} f(x+k) = f(x+) = g(x)$

Which shows that g is right-continuous.