Solutions to HW#4

1. Yes, f is increasing. Let $y > x$. Set $a_n = f_n(y) - f_n(x)$. Then $a_n \geq 0$ for all n and therefore $\lim_{n\to\infty} a_n = f(y) - f(x) \geq 0$. Hence $f(y) \geq f(x)$.

2. Before we get to the problem at hand, it would be useful to set our notation. Recall that a sequence $\{x_n\}_{n=1}^{\infty}$ with elements in the set X is some function $\varphi : \mathbb{N} \mapsto X$. A subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ is then just a composition $\varphi \circ \sigma : \mathbb{N} \mapsto X$ where $\sigma : \mathbb{N} \mapsto \mathbb{N}$ is a strictly increasing function. Similarly, a subsequence of a subsequence is a composition of the form $\varphi \circ \sigma \circ \tau : \mathbb{N} \mapsto X$ where $\sigma, \tau : \mathbb{N} \mapsto \mathbb{N}$ are strictly increasing functions.

In this problem we will have to "jump" through layers of subsequences. Instead of writing $\{x_{n_p}\}_{p=1}^{\infty}$, $\{x_{n_{p_L}}\}_{L=1}^{\infty}$, and $\{x_{n_{p_{L_K}}}\}_{K=1}^{\infty}$ for the first, second, and third layer, we'll simply write $\{x_{\sigma_1(n)}\}_{n=1}^{\infty}$, $\{x_{\sigma_2(n)}\}_{n=1}^{\infty}$, $\{x_{\sigma_3(n)}\}_{n=1}^{\infty}$ respectfully.

Now to the problem: Let $\{r_m\}_{m=1}^{\infty}$ be an enumeration of the rationals in $[a,b]$. Since $|f_n(x)| \leq 1$ for all x and n, $\{f_n(r_1)\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$. Recall that every bounded sequence in $\mathbb{R}$ contains a convergent subsequence $\{f_{\sigma_1(n)}\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} f_{\sigma_1(n)}(r_1) = L_1$.

Consider the sequence $\{f_{\sigma_1(n)}(r_2)\}_{n=1}^{\infty}$. This sequence is also bounded and therefore has a convergent subsequence $\{f_{\sigma_2(n)}(r_2)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} f_{\sigma_2(n)}(r_2) = L_2$. Notice that since every convergent sequence has only convergent subsequences, $\lim_{n\to\infty} f_{\sigma_2(n)}(r_1) = L_1$. Set $\sigma_2^*: \mathbb{N} \to \mathbb{N}$ as

$$\sigma_2^*(n) = \begin{cases} \sigma_1(1) & \text{if } n=1 \\ \sigma_2(n) & \text{otherwise} \end{cases}$$

Then $\lim_{n\to\infty} f_{\sigma_2^*(n)}(r_1) = L_1$ and $\lim_{n\to\infty} f_{\sigma_2^*(n)}(r_2) = L_2$.

Now find a subsequence $\{f_{\sigma_3(n)}\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} f_{\sigma_3(n)}(r_3) = L_3$. Continue in this fashion to obtain a subsequence $\{f_{\sigma(n)}\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} f_{\sigma(n)}(r_m) = L_m$ for all $m \in \mathbb{N}$. Here is the process:

$$\begin{array}{llllll}
\boxed{f_{\sigma_1(1)}}, & f_{\sigma_1(2)}, & f_{\sigma_1(3)}, & f_{\sigma_1(4)}, & f_{\sigma_1(5)}, & \ldots \\
f_{\sigma_2(1)}, & \boxed{f_{\sigma_2(2)}}, & f_{\sigma_2(3)}, & f_{\sigma_2(4)}, & f_{\sigma_2(5)}, & \ldots \\
f_{\sigma_3(1)}, & f_{\sigma_3(2)}, & \boxed{f_{\sigma_3(3)}}, & f_{\sigma_3(4)}, & f_{\sigma_3(5)}, & \ldots \\
f_{\sigma_4(1)}, & f_{\sigma_4(2)}, & f_{\sigma_4(3)}, & \boxed{f_{\sigma_4(4)}}, & f_{\sigma_4(5)} & \ldots \\
f_{\sigma_5(1)}, & f_{\sigma_5(2)}, & f_{\sigma_5(3)}, & f_{\sigma_5(4)}, & \boxed{f_{\sigma_5(5)}} & \ldots
\end{array}$$

In particular

$$\sigma(n) = \sigma_n(n).$$

3. Suppose $f_n \overset{i}{\rightrightarrows} f$ & $g_n \overset{i}{\rightrightarrows} g$. Let n_0 be large enough to satisfy $\|f_n - f\|_\infty < \frac{\epsilon}{2}$ and $\|g_n - g\|_\infty < \frac{\epsilon}{2}$ for all $n \geq n_0$. Then for any $x \in X$, $|f_n(x) + g_n(x) - f(x) - g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$ $\leq \|f_n - f\|_\infty + \|g_n - g\|_\infty < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.
It follows that $\|f_n + g_n - (f+g)\|_\infty \leq \epsilon$ or $f_n + g_n \overset{i}{\rightrightarrows} f+g$.
To see that uniform convergence is not generally preserved by products, notice that if $X = \mathbb{R}$ and $f_n(x) = x + \frac{1}{n}$, $g_n(x) = x^2 - \frac{1}{n}$ then $f_n \overset{i}{\rightrightarrows} f$ & $g_n \overset{i}{\rightrightarrows} g$, where $f(x) = x$ and $g(x) = x^2$.
$f_n g_n(x) = (x + \frac{1}{n})(x^2 - \frac{1}{n}) = x^3 + \frac{1}{n}(x + x^2) - \frac{1}{n^2}$ and $fg(x) = x^3$.
Notice that $\|f_n g_n - fg\|_\infty = \infty$ for all n, since

$$\lim_{x\to\infty} |f_n g_n(x) - fg(x)| = \lim_{x\to\infty} \frac{1}{n}|x^2 - x - \frac{1}{n}| = \infty.$$

4. No. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \begin{cases} 0 & \text{if } x \in [-n, n] \\ 1 & \text{otherwise} \end{cases}$
Then $f_n \overset{i}{\rightrightarrows} 0$ on every closed and bounded interval. However, f_n converges to 0 pointwise only on $\mathbb{R}$.
5. For $\epsilon > 0$, let $\delta > 0$ be small enough to satisfy $|f(y) - f(x)| < \epsilon$ for all $x, y \in \mathbb{R}$ such that $|y - x| < \delta$. Let $N \in \mathbb{N}$ be such that $\frac{1}{n} < \delta$ for all $n \geq N$. Then $|f_n(x) - f(x)| = |f(x + \frac{1}{n}) - f(x)| < \epsilon$ for all x and all $n \geq N$. Hence $\|f_n - f\|_\infty \leq \epsilon$. It follows that $f_n \overset{i}{\rightrightarrows} f$.
6. First observe that, since $f_n \overset{i}{\rightrightarrows} f$ and f_n is continuous at x for all n, f must also be continuous at x:

Let $n_0 \in \mathbb{N}$ be such that $\rho(f_n(y), f(y)) < \frac{\epsilon}{3}$ for all $y \in X$, $n \geq n_0$.
Now set $d(y,x) < \delta$ so $\rho(f_{n_0}(y), f_{n_0}(x)) < \frac{\epsilon}{3}$. Then

$$\rho(f(x), f(y)) < \rho(f(x), f_{n_0}(x)) + \rho(f_{n_0}(x), f_{n_0}(y)) + \rho(f_{n_0}(y), f(y)) < \epsilon.$$

Now if $x_n \overset{d}{\mapsto} x$, there is some $N \in \mathbb{N}$ for which $\rho(f(x_n), f(x)) < \frac{\epsilon}{2}$ whenever $n \geq N$. Then for $n \geq \max\{n_0, N\}$

$$\rho(f_n(x_n), f(x)) \leq \rho(f_n(x_n), f(x_n)) + \rho(f(x_n), f(x)) < \frac{\epsilon}{3} + \frac{\epsilon}{2} < \epsilon.$$

7. Suppose X is compact, $f_n, f \in C(X)$, $f_n \mapsto f$, and $f_n(x) \leq f_{n+1}(x)$ for each n and $x \in X$. For $\epsilon > 0$ define $\mathcal{U}_n = \{x \in X : f(x) - f_n(x) < \epsilon\}$. Then, since $f(x) - f_{n+1}(x) = |f(x) - f_{n+1}(x)| \leq |f(x) - f_n(x)| = f(x) - f_n(x)$, $\mathcal{U}_n \subset \mathcal{U}_{n+1}$ for each n. Since f and f_n are continuous, each $\mathcal{U}_n$ is an open subset of X (Why?). Moreover, since $f_n \mapsto f$, each $x \in X$ is eventually in some $\mathcal{U}_n$. That is, $\lim_{n \to \infty} \mathcal{U}_n = X$. It follows that $\{\mathcal{U}_n : n \geq 1\}$ is an open cover of X. Since X is compact, there is a finite subcover $\{\mathcal{U}_{n_1}, \dots, \mathcal{U}_{n_k}\}$ of X. But $X \subset \bigcup_{i=1}^{k} \mathcal{U}_{n_i} = \mathcal{U}_{n_k}$. That is, for all $x \in X$, $|f(x) - f_{n_k}(x)| < \epsilon$.
If $m \geq n_k$, $|f(x) - f_m(x)| \leq |f(x) - f_{n_k}(x)| < \epsilon$. Hence $\|f - f_m\|_\infty \leq \epsilon$ for all $m \geq n_k$. In particular $f_n \rightrightarrows f$.

8. This is true. Since $f_n \rightrightarrows f$, $\{f_n\}$ is Cauchy. Thus there is some number $B > 0$ such that $\|f_n\|_\infty \leq B$ for all $n \in \mathbb{N}$. Also, it is clear that $\|f_n - f\|_\infty \mapsto 0$ as $n \mapsto \infty$.

$$\left|\int_0^{1-\frac{1}{n}} f_n - \int_0^1 f\right| \leq \left|\int_0^{1-\frac{1}{n}} f_n - \int_0^1 f_n\right| + \left|\int_0^1 f_n - \int_0^1 f\right| =$$

$$= \left|\int_{1-\frac{1}{n}}^1 f_n\right| + \left|\int_0^1 (f_n - f)\right| \leq \int_{1-\frac{1}{n}}^1 |f_n| + \int_0^1 |f_n - f| \leq$$

$$\leq \left[1-\left(1-\tfrac{1}{n}\right)\right]\|f_n\|_\infty + \|f_n - f\|_\infty \leq \tfrac{1}{n}B + \|f_n - f\|_\infty \longmapsto 0 \text{ as } n \longmapsto \infty.$$

In other words, $\int_0^{1-\frac{1}{n}} f_n \mapsto \int_0^1 f_n$.

9, Let $f, g \in B(X)$. Define $A = \{f(x)g(x) : x \in X\}$ and

$\mathcal{B} = \{f(x)g(y) : x, y \in X\}$. Then $A \subset \mathcal{B}$ and $\sup \mathcal{B} = \|f\|_\infty \|g\|_\infty$

Hence $\sup A = \sup_{x \in X} |f(x)g(x)| = \|fg\|_\infty \leq \sup \mathcal{B} = \|f\|_\infty \|g\|_\infty$.

In particular, $fg \in B(X)$.

Suppose now that $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$. Then $f, g \in B(X)$ and for any fixed $x \in X$ we have

$$|f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x)g_n(x) - f(x)g_n(x)| + |f(x)g_n(x) - f(x)g(x)|$$

$$= |g_n(x)||f_n(x) - f(x)| + |f(x)||g_n(x) - g(x)| \leq C\|f_n - f\|_\infty + \|f\|_\infty \|g_n - g\|_\infty$$

where $C > 0$ is an upper bound of $\|g_n\|_\infty$ for all $n \in \mathbb{N}$. This upper bound exists because $\{g_n\}$ is uniformly Cauchy (i.e. Cauchy in $B(X)$).

Since $x \in X$ was arbitrarily chosen, we see that

$$\|f_n g_n - fg\|_\infty \leq C\|f_n - f\|_\infty + \|f\|_\infty \|g_n - g\|_\infty \longmapsto 0 \text{ as } n \longmapsto \infty$$

In particular, $f_n g_n \rightrightarrows fg$.

10. Let $f(x) = \sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$. Define $f_N(x) = \sum_{n=1}^{N} \frac{x^2}{(1+x^2)^n}$.

That is $f(x) = \lim_{N\to\infty} f_N(x)$. Notice that $f_N(0) = 0$ for all N

so $f(0) = \lim_{N\to 0} f_N(0) = 0$ for $x \neq 0$,

$$\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = \frac{x^2}{1+x^2} \sum_{n=1}^{\infty} \left(\frac{1}{1+x^2}\right)^{n-1} = \frac{x^2}{1+x^2} \quad \frac{1+x^2}{x^2} = 1$$

Hence $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$ is not continuous,

despite its being the limit of continuous functions. Since continuity is preserved by uniform convergence, we see that f_N does not converge to f uniformly.

11. Since $\{f_n\}$ converges uniformly on $\mathbb{Q}$, $\{f_n\}$ is uniformly Cauchy on $\mathbb{Q}$. That is, for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $\sup_{r\in\mathbb{Q}} |f_n(r) - f_m(r)| < \epsilon/3$ for all $r \in \mathbb{Q}$ whenever $n, m \geq N$.

Since f_n is continuous on $\mathbb{R}$ for all n, for any $x \in \mathbb{R}$ and any $n, m \in \mathbb{N}$ we can select $r \in \mathbb{Q}$ close enough to x so that $|f_n(x) - f_n(r)| < \epsilon/3$ and $|f_m(x) - f_m(r)| < \epsilon/3$.

Then $|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(r)| + |f_n(r) - f_m(r)| + |f_m(r) - f_m(x)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$ whenever $n, m \geq N$. (For different pairs n, m, different choices of r must be made)

It follows that $\{f_n\}$ is uniformly Cauchy on $\mathbb{R}$. Hence $\{f_n\} \subset C_b(\mathbb{R})$ and therefore $f_n \rightrightarrows f \in C_b(\mathbb{R})$.

12. (a) Let $f_N(x) = \sum_{n=1}^{N} ne^{-nx}$ and $f(x) = \lim_{N\mapsto\infty} f_N(x) = \sum_{n=1}^{\infty} ne^{-nx}$

For $x \leq 0$, $f_N(x) \mapsto \infty$ as $N \mapsto \infty$. If we consider

$f_N : [\delta, \infty) \mapsto \mathbb{R}$ for any $\delta > 0$, the M-test yields

$$\sum_{n=1}^{\infty} \| ne^{-nx} \|_\infty = \sum_{n=1}^{\infty} ne^{-n\delta} < \infty \text{ so } f_N \overset{u}{\mapsto} f \text{ on } [\delta, \infty).$$

Clearly $f_N \mapsto f$ on $(0, \infty)$. On $(0, \infty)$ convergence is not uniform.

(b) $f_N \overset{u}{\mapsto} f$ on $[1, 2]$. Therefore

$$\int_1^2 \sum_{n=1}^{\infty} ne^{-nx} dx = \int_1^2 \lim_{N\mapsto\infty} f_N(x)\, dx = \lim_{N\mapsto\infty} \int_1^2 f_N(x)\, dx =$$

$$= \lim_{N\mapsto\infty} \int_1^2 \sum_{n=1}^{N} ne^{-nx} dx = \lim_{N\mapsto\infty} \sum_{n=1}^{N} \int_1^2 ne^{-nx} dx = \sum_{n=1}^{\infty} \int_1^2 ne^{-nx} dx =$$

$$= \sum_{n=1}^{\infty} -e^{-nx} \Big|_1^2 = \sum_{n=1}^{\infty} [e^{-n} - e^{-2n}] = \frac{e^{-1}}{1-e^{-1}} - \frac{e^{-2}}{1-e^{-2}} =$$

$$= \frac{e}{e^2-e} - \frac{1}{e^2-1} = \frac{e}{e^2-1}$$

13. Let $f_n(x) = \dfrac{x}{n^\alpha(1+nx^2)}$. Then $|f_n(x)| = \dfrac{|x|}{n^\alpha(1+nx^2)} =$

$$= \frac{\sqrt{n}\,|x|}{n^{\alpha+1/2}(1+[\sqrt{n}\,|x|]^2)} = \frac{\sqrt{n}\,|x|}{1+[\sqrt{n}\,|x|]^2} \cdot \frac{1}{n^{\alpha+1/2}}.$$

Let $u = \sqrt{n}\,|x|$ and consider $g(u) = \dfrac{u}{1+u^2}$. Then g attains a maximum value at $u = 1$. Hence $|f_n|$ attains its max at $|x| = 1/\sqrt{n}$. This value is $\|f_n\|_\infty = \dfrac{1}{1+1} \cdot \dfrac{1}{n^{\alpha+1/2}} = \dfrac{1}{2n^{\alpha+1/2}}$

Thus, provided that $\alpha > \frac{1}{2}$, the M-test implies

$$\sum_{n=1}^{\infty} \|f_n\|_\infty \leq \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1/2}} < \infty \quad \text{so} \quad \sum_{n=1}^{\infty} f_n \text{ converges uniformly}$$

on any bounded interval of $\mathbb{R}$.

14. Let $f_N(x) = \sum_{n=1}^{N} \frac{nx^2}{n^3+x^2}$ and $f(x) = \sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^2}$.

If $a > 0$, $f_N : [0, a] \mapsto \mathbb{R}$ describes a continuous function. By the M-test, $\sum_{n=1}^{\infty} \left\| \frac{nx^2}{n^3+x^2} \right\|_\infty \leq \sum_{n=1}^{\infty} \frac{na^2}{n^3} = a^2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

so $f_N \overset{\mapsto}{\mapsto} f$ on $[0, a]$ where f must be continuous on $[0, a]$. In particular, f is continuous on $[0, 2]$. Thus,

$$f(1) = \lim_{x \to 1} f(x) = \lim_{x \to 1} \lim_{N \mapsto \infty} f_N(x)$$

$$f(1) = f(\lim_{x \to 1} x) = \lim_{N \to \infty} f_N(\lim_{x \to 1} x) = \lim_{N \mapsto \infty} \lim_{x \to 1} f_N(x).$$

In particular, $f(1) = \lim_{N \mapsto \infty} f_N(1) = \lim_{N \mapsto \infty} \sum_{n=1}^{N} \frac{n}{n^3+1} =$

$$= \sum_{n=1}^{\infty} \frac{n}{n^3+1}$$

15. (a) If $\sum_{n=1}^{\infty} |a_n| < \infty$. Consider $f_N, f : [0, \infty)$ defined by $f_N(x) = \sum_{n=1}^{N} a_n e^{-nx}$ and $f(x) = \sum_{n=1}^{\infty} a_n e^{-nx}$. Then

$$\sum_{n=1}^{\infty} \|a_n e^{-nx}\|_\infty = \sum_{n=1}^{\infty} |a_n| < \infty.$$ Hence $f_N \overset{\mapsto}{\mapsto} f$ by the M-test.

(b) If $\{a_n\}_{n=1}^{\infty}$ is bounded, there is some number $c>0$ such that $|a_n| \leq c$. If we assume that f_N, f are defined on the interval $[\delta, \infty)$, where $\delta > 0$, we get

$$\sum_{n=1}^{\infty} \|a_n e^{-nx}\|_{\infty} = \sum_{n=1}^{\infty} |a_n| e^{-n\delta} \leq \sum_{n=1}^{\infty} c e^{-n\delta} < \infty$$

Hence $f_N \rightrightarrows f$ on $[\delta, \infty)$ by the M-test.

16. Let $f = \sum_{n=1}^{\infty} g_n$ and $f_N = \sum_{n=1}^{N} g_n$ be defined on $[a,b]$. By hypothesis, f and g_n are continuous with $0 \leq g_n$. Hence f_N is continuous for each N and $f_N(x) \leq f_{N+1}(x)$ for any $x \in [a,b]$. Since $[a,b]$ is compact and $f_N \to f$, we apply Dini's theorem to conclude that $f_N \rightrightarrows f$.

17. Suppose $\{f_n\}_{n=1}^{\infty} \subset C(0,\infty)$ with $|f_n(x)| \leq n$ for every $x > 0$. Consider $f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x)$. Applying the M-test, we conclude that $\sum_{n=1}^{\infty} \|2^{-n} f_n(x)\|_{\infty} \leq \sum_{n=1}^{\infty} 2^{-n} n < \infty$. This means that $g_N = \sum_{n=1}^{N} 2^{-n} f_n \rightrightarrows f$. In particular, since each g_N is continuous, f is continuous on $(0,\infty)$. By hypothesis $\lim_{x\to\infty} f_n(x) = 0$ for each n. Thus $\lim_{x\to\infty} \sum_{n=1}^{N} 2^{-n} f_n(x) = 0$.

In particular, $\lim_{x\to\infty} g_N(x) = 0$ for each $N \in \mathbb{N}$.

Now, since $g_N \rightrightarrows f$ on $(0,\infty)$, there exists some $K \in \mathbb{N}$ such that $\|f - g_N\|_\infty < \epsilon/2$ for all $N \geq K$.

Hence $|f(x)| \leq |f(x) - g_N(x)| + |g_N(x)| < \epsilon$ for $N \geq K$ and x large enough to satisfy $|g_N(x)| < \frac{\epsilon}{2}$.

In particular, $\lim_{x\to\infty} |f(x)| = 0$.