Solutions to H.W. # 3

1. If K is a compact subset of $\mathbb{R}$, then K is bounded and closed. Since K is bounded, $\sup K$ and $\inf K$ are elements of $\mathbb{R}$. Notice that $\sup K$ is either an element of K or a limit point of K (or both). Since K is closed, K must contain all of its limit points. In particular, $\sup K \in K$. Similarly, $\inf K \in K$.

2. Notice that $E = \left([-\sqrt{3}, -\sqrt{2}] \cap \mathbb{Q}\right) \cup \left([\sqrt{2}, \sqrt{3}] \cap \mathbb{Q}\right)$.
Hence E is closed and bounded in $\mathbb{Q}$ (Why?).
Observe that E is not complete, because $\sqrt{2} \notin \mathbb{Q}$ is a limit point of E (which means that some Cauchy sequence of E fails to converge in E) Thus E is not compact.

3. Let A be a compact subset of (M, d) and B be a compact subset of (N, ρ). Then $(M \times N, \sigma)$ is a metric space with metric function $\sigma : (M \times N)^2 \mapsto \mathbb{R}$ defined by

$$\sigma((x,y),(z,w)) = d(x,z) + \rho(y,w)$$

We will show that $A \times B$ is compact in $M \times N$ by proving that $A \times B$ is complete and totally bounded.
To see that $A \times B$ is complete, pick a Cauchy sequence $\{(x_n, y_n)\}_{n=1}^{\infty} \subset A \times B$. Then, in particular, $\{x_n\}_{n=1}^{\infty} \subset A$ and

$\{y_n\}_{n=1}^{\infty} \subset B$ are Cauchy, (Why?) Thus $x_n \overset{d}{\longmapsto} x \in A$ and $y_n \overset{\rho}{\longmapsto} y \in B$ (because A and B are complete). Notice that $d(x_n, x) \longmapsto 0$ and $\rho(y_n, y) \longmapsto 0$ implies that $\sigma((x_n, y_n), (x, y)) \longmapsto 0$. Thus $(x_n, y_n) \overset{\sigma}{\longmapsto} (x, y) \in A \times B$. Hence every Cauchy sequence in $A \times B$ converges in $A \times B$.

To prove that $A \times B$ is totally bounded, note that for $\epsilon > 0$,

$A = \bigcup_{i=1}^{n} A_i$ where $\text{diam}(A_i) < \frac{\epsilon}{2}$ for all $i \in \{1, \dots, n\}$ and

$B = \bigcup_{j=1}^{m} B_j$ where $\text{diam}(B_j) < \frac{\epsilon}{2}$ for all $j \in \{1, \dots, m\}$. In other words, because A and B are totally bounded, they can be broken up into finitely many sets, each of diameter less than $\epsilon/2$. But this means that $A \times B = \bigcup_{i,j} A_i \times B_j$ is the union of nm sets $A_i \times B_j$ sets with

$$\text{diam}(A_i \times B_j) = \sup_{(x,y),(z,w) \in A_i \times B_j} \sigma((x,y),(z,w)) = \sup_{x,z \in A_i;\, y,w \in B_j} \{d(x,z) + \rho(y,w)\} \leq$$

$$\leq \sup_{x,z \in A_i} d(x,z) + \sup_{y,w \in B_j} \rho(y,w) = \text{diam}(A_i) + \text{diam}(B_j) < \epsilon$$

Thus $A \times B$ is totally bounded.

4. If A is compact in M then, by problem 3, $A \times A$ is compact in $(M \times M, \sigma)$, where $\sigma((x,y),(a,b)) = d(x,a) + d(y,b)$.

Define $f: M\times M \mapsto \mathbb{R}$ by $f(x,y) = d(x,y)$. Then f is Lipschitz:

$$|f(x,y) - f(a,b)| = |d(x,y) - d(a,b)| =$$
$$= |d(x,y) - d(a,y) + d(a,y) - d(a,b)| \leq$$
$$\leq |d(x,y) - d(a,y)| + |d(a,y) - d(a,b)| \leq$$
$$\leq d(x,a) + d(y,b) = \mathcal{O}((x,y),(a,b)).$$

In particular, f is continuous.

Now $\operatorname{diam}(A) = \sup\{d(x,y) : x,y\in A\} = \sup f(A\times A)$. But $f(A\times A)$ is compact since it is the image of a compact set under a continuous function. Hence, by problem 1, $\sup f(A\times A)$ is an element of $f(A\times A)$. In other words, there must be a point $(x,y)\in A\times A$ such that $\sup(A\times A) = f(x,y) = d(x,y)$.

5. Let $\{c_n\}_{n=1}^{\infty}$ be a nonnegative sequence of real numbers. Define $K = \{x\in l_2 : |x_n| \leq c_n,\ n\geq 1\}$. Suppose that $\{c_n\}_{n=1}^{\infty}\in l_2$. Then $\sum_{n=1}^{\infty} c_n^2 < \infty$. We will show that K is compact.

<u>$K\subset l_2$ is closed</u>: let $\{f_n\}_{n=1}^{\infty}\subset K$ be Cauchy. Then $f_n \mapsto f$ where $f\in l_2$. Notice that $|f(k)| = |\lim_{n\to\infty} f_n(k)| = \lim_{n\to\infty} |f_n(k)| \leq c_k$ for all $k\in\mathbb{N}$. This implies that $f\in K$. In particular, K is closed and hence complete subset of l_2.

<u>K is totally bounded:</u> For every $\epsilon > 0$, let $N \in \mathbb{N}$ be such that $\sum_{n=N+1}^{\infty} c_n^2 < \epsilon^2$. Define $A_N = \{x \in K : |x_n| = 0 \text{ for all } n \geq N\}$.

Notice that if $x \in A_N$, then $x = (x_1, \dots, x_N, 0, 0, \dots)$ where $|x_n| \leq c_n$. Thus A_N is isometric to a bounded subset of $\mathbb{R}^N$. In particular, A_N is totally bounded.

Observe that any element $x = (x_1, \dots, x_N, x_{N+1}, \dots) \in K$ is within a distance of $\epsilon > 0$ from $y = (x_1, \dots, x_N, 0, \dots) \in A_N$, since

$$\|x - y\|_2 = \sqrt{\sum_{n=N+1}^{\infty} x_n^2} \leq \sqrt{\sum_{n=N+1}^{\infty} c_n^2} = \sqrt{\epsilon^2} = \epsilon$$

Since A_N is totally bounded, we can pick $y_1, \dots, y_m \in A_N$ such that $A_N \subset \bigcup_{i=1}^{m} B_\epsilon(y_i)$. We will prove that $K \subset \bigcup_{i=1}^{m} B_{2\epsilon}(y_i)$.

If $x \in K$, then there is some $y \in A_N$ such that $\|x - y\|_2 < \epsilon$. This y must be in some $B_\epsilon(y_i)$. Thus $\|x - y_i\|_2 \leq$ $\leq \|x - y\|_2 + \|y - y_i\|_2 < \epsilon + \epsilon = 2\epsilon$. Hence $x \in B_{2\epsilon}(y_i)$. In other words, every $x \in K$ is in some ball $B_{2\epsilon}(y_i)$ and the desired result follows.

Suppose now that $\{c_n\}_{n=1}^{\infty} \notin l_2$. Then $\sum_{n=1}^{\infty} c_n^2 = \infty$. Notice that for every $M > 0$ there is some $N \in \mathbb{N}$ such that $\sum_{n=1}^{N} c_n^2 > M^2$. Then $c = (c_1, \dots, c_N, 0, 0, \dots) \in K$ and $\|c\|_2 > M$. This shows that K is not bounded and therefore not compact.

6. If E is a noncompact subset of $\mathbb{R}$, then E is not closed or E is not bounded.

(a) <u>E is not closed</u>: E does not contain all of its limit points. Let y be a limit point of E that is not in E. Then

(i) The function $f: E \mapsto \mathbb{R}$ given by $f(x) = \frac{1}{d(x,y)}$ is continuous, but not bounded (why?)

(ii) The function $f: E \mapsto \mathbb{R}$ given by $f(x) = \dfrac{\frac{1}{d(x,y)}}{1 + \frac{1}{d(x,y)}}$

is continuous and bounded, but f fails to attain a maximum value on E.

(b) <u>E is not bounded</u>:

(i) The function $f: E \mapsto \mathbb{R}$ given by $f(x) = d(x,a)$ for any fixed $a \in E$ is continuous and unbounded.

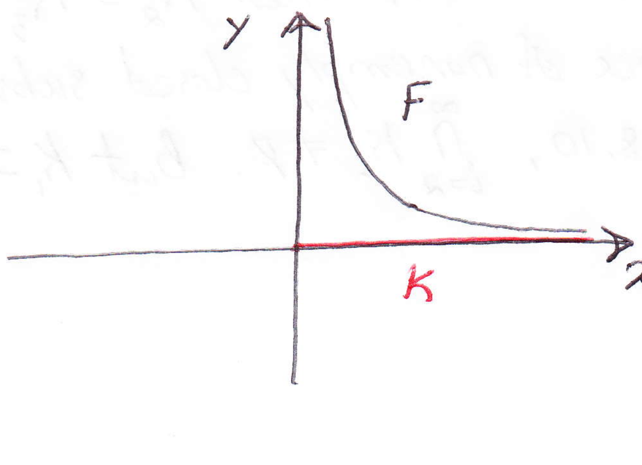
(ii) The function $f: E \mapsto \mathbb{R}$ given by $f(x) = \dfrac{d(x,a)}{1+d(x,a)}$ for any fixed $a \in E$ is continuous, bounded, but without max.

7. Let $K_1 \supset K_2 \supset \ldots \supset K_n \supset \ldots$ be a decreasing sequence of nonempty compact subsets of M. Then $K_2 \supset K_3 \supset \ldots \supset K_n \supset \ldots$ is a decreasing sequence of nonempty closed subsets of K_1. According to Corollary 8.10, $\bigcap_{i=2}^{\infty} K_i \neq \emptyset$. But $K_1 \supset \bigcap_{i=2}^{\infty} K_i$ so $\bigcap_{i=1}^{\infty} K_i = \bigcap_{i=2}^{\infty} K_i \neq \emptyset$.

8. Suppose that F is closed and K is compact in M and that $d(F,K)=0$. Then there must be a sequence $\{(x_n,y_n)\}_{n=1}^{\infty}$ for which $d(x_n,y_n)\mapsto 0$. Since $\{y_n\}_{n=1}^{\infty}$ is a sequence in K, there is some subsequence $y_{n_k} \overset{d}{\mapsto} y \in K$ (why?). The corresponding subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of F still satisfies $d(x_{n_k},y_{n_k})\mapsto 0$ as $k\mapsto\infty$, which means that $x_{n_k} \overset{d}{\mapsto} y$ as well: $d(x_{n_k},y)\leq d(x_{n_k},y_{n_k})+d(y_{n_k},y)$ $\mapsto 0$ as $k\mapsto\infty$. Thus, since F is closed and y is its limit point, we see that $y\in F$. Hence $F\cap K\neq\emptyset$.

We have established that if $d(F,K)=0$, then $F\cap K\neq\emptyset$ under the assumption that F is closed and K is compact. Notice that if K is merely closed, we can have $d(F,K)=0$ even when $F\cap K=\emptyset$:

Let $F=\{(x,1/x): x>0\}$ and $K=\{(x,0): x\geq 0\}$ be subsets of $\mathbb{R}^2$. Then F and K are nonempty, disjoint closed subsets of $\mathbb{R}^2$ with $d(F,K)=0$.



9. Suppose $f:(M,d) \mapsto (N,\rho)$ is Lipschitz. Then for $\epsilon > 0$ set $\delta(\epsilon) = \frac{\epsilon}{K}$. Note that when $d(x,y) < \delta(\epsilon)$ for any $x, y \in M$, $\rho(f(x), f(y)) \leq K\,d(x,y) < K\delta(\epsilon) = \epsilon$. Thus f is uniformly continuous.

10. Let $f:(M,d) \mapsto (N,\rho)$ be uniformly continuous and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in M. We wish to show that $\{f(x_n)\}_{n=1}^{\infty}$ must be a Cauchy sequence in N. To do so, let $\epsilon > 0$. Pick $\delta > 0$ such that $\rho(f(x), f(y)) < \epsilon$ whenever $d(x,y) < \delta$. Since $\{x_n\}_{n=1}^{\infty}$ is Cauchy, there is some $N \in \mathbb{N}$ such that $d(x_n, x_m) < \delta$ for all $n, m \geq N$. But then $\rho(f(x_n), f(x_m)) < \epsilon$. Hence the terms of the sequence $\{f(x_n)\}_{n=1}^{\infty}$ cluster together as $n \mapsto \infty$, showing that $\{f(x_n)\}_{n=1}^{\infty}$ is indeed Cauchy.

11. Suppose that $f:(0,1) \mapsto \mathbb{R}$ is uniformly continuous and let $\{x_n\}_{n=1}^{\infty}$ be any sequence in $(0,1)$ that converges to 0 (in $\mathbb{R}$). Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(0,1)$. By the result in problem 10, we know that $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Hence $\{f(x_n)\}_{n=1}^{\infty}$ converges (to something). In particular, given any arbitrary sequence in $(0,1)$ that converges to 0, the image of this sequence under f must converge (to something!).

But this means that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$$

whenever $x_n \mapsto 0$ and $y_n \mapsto 0$, $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subset (0,1)$ (why?) Thus $\lim_{x\mapsto 0^+} f(x)$ exists as desired. Using the same argument, we see that $\lim_{x\mapsto 1^-} f(x)$ exists as well. Finally, to see that $f((0,1))$ is bounded, define $g: [0,1] \mapsto \mathbb{R}$ by $g(x) = f(x)$ for $x \in (0,1)$, $g(0) = f(0+)$, and $g(1) = f(1-)$. Clearly then, g is continuous and $g([0,1])$ is bounded (why?). Hence $f((0,1)) = g((0,1)) \subset g([0,1])$ is bounded as well.

12. Let $f: \mathbb{R} \mapsto \mathbb{R}$ be given by $f(x) = \sin(2\pi x^2)$. Then f is continuous and bounded ($f(\mathbb{R}) = [-1,1]$), but f is not uniformly continuous. In particular, if $0 < \epsilon < 1$, there is no $\delta > 0$ such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$. To see this, observe first that $\lim_{n\to\infty} (\sqrt{n} - \sqrt{n-1}) = 0$. Hence, for any fixed $\delta > 0$, there is some n, such that $\sqrt{n} - \sqrt{n-1} < \delta$. Let $x = \sqrt{n}$ and $y = \sqrt{n-1}$. Then in the interval $[\sqrt{n-1}, \sqrt{n}]$ there is a pair of points $w, z \in (\sqrt{n-1}, \sqrt{n})$ such that $f(w) = 1$ and $f(z) = -1$. But then $|w - z| < |x-y| < \delta$, while

$|f(w) - f(z)| = 2 > \epsilon$.

Notice, however, that an unbounded continuous function may be uniformly continuous (and even Lipschitz).
Let $g: \mathbb{R} \mapsto \mathbb{R}$ be given by $g(x) = -3x + 1$.
Then $|g(x) - g(y)| = 3|x-y|$. Hence g is Lipschitz.
Clearly $g(\mathbb{R}) = \mathbb{R}$ so g is not bounded.

13. Suppose $f: M \mapsto N$ is uniformly continuous and $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty} \subset M$ are such that $\rho(f(x), f(y)) < \epsilon$ whenever $d(x,y) < \delta$. Then, for some $N > 0$, $d(x_n, y_n) < \delta$ for all $n \geq N$ and consequently $\rho(f(x_n), f(y_n)) < \epsilon$. Hence $\rho(f(x_n), f(y_n)) \mapsto 0$.
Suppose now that f is not uniformly continuous. Then, for some $\epsilon > 0$, there is no $\delta > 0$ such that $\rho(f(x), f(y)) < \epsilon$ for all $x, y \in M$ satisfying $d(x,y) < \delta$. This implies that there is a pair of sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty} \subset M$ for which $d(x_n, y_n) \mapsto 0$, but $\rho(f(x_n), f(y_n)) \geq \epsilon$ for all n:
For $\delta = \frac{1}{n}$, choose a pair of points $x_n, y_n \in M$ such that $d(x_n, y_n) < \frac{1}{n}$, while $\rho(f(x_n), f(y_n)) \geq \epsilon$. Such a pair always exists, because $\delta = \frac{1}{n}$ does not "work" for every pair $x, y \in M$.

14. Suppose f is differentiable at $x = a$. Then, for $\epsilon > 0$, there is a $\delta > 0$ such that $|F(x) - f'(a)| < \epsilon/2$ for all x

satisfying $0 < |x-a| < \delta$. Let $x, y \in (a-\delta, a) \cup (a, a+\delta)$.
Then $|F(x) - F(y)| \leq |F(x) - f'(a)| + |f'(a) - F(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.
Thus, when f is differentiable at $x = a$, the function
$F(x) = \dfrac{f(x)-f(a)}{x-a}$ is uniformly continuous in some punctured neighborhood of a.

Suppose now that F is uniformly continuous in some punctured neighborhood of a. Then $\lim\limits_{x \to a} F(x) = \lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a}$ must exist (Why? look at problem 11 carefully!)

15. Simply set $\delta(\epsilon) = \left(\frac{\epsilon}{K}\right)^{1/\alpha}$.

16. Suppose that $f: \mathbb{R} \mapsto \mathbb{R}$ has a bounded derivative, $\|f'\|_\infty \leq K$. Then $\dfrac{|f(x)-f(y)|}{|x-y|} = \left|\dfrac{f(x)-f(y)}{x-y}\right| = |f'(c)| \leq K$;

where c is some number in (x, y).
Hence $|f(x) - f(y)| \leq K|x-y|^1$ is Lipschitz of order 1.

17. Suppose $|f(x) - f(y)| \leq K|x-y|^\alpha$ for all $x, y \in \mathbb{R}$ and $\alpha > 1$. Then for any $a \in \mathbb{R}$, $0 \leq \lim\limits_{x \to a} \dfrac{|f(x)-f(a)|}{|x-a|} \leq$

$\leq \lim\limits_{x \to a} K|x-a|^{\alpha-1} = 0$. Thus $f'(a) = 0$ for all $a \in \mathbb{R}$. It follows that f is constant.