Solutions to H.W#1

1. Let E be connected and suppose $E \subset A \cup B$ where A and B are open and disjoint subsets of the metric (M,d). If $E \cap A \neq \emptyset$ then $E \subset A$. Otherwise $E \cap B \neq \emptyset$ as well, which would make $\{A, B\}$ a disconnection of E. By hypothesis, no such disconnection is possible.

2. Let $E = A \cup B \subset M$, $A \cap B = \emptyset$. Then the sets A and B are clopen relative to E if and only if both A and B are closed relative to E, which happens if and only if A contains no limit points in B and B contains no limit points in A. But the assertion that A has no limit points in B is the same as $\overline{A} \cap B = \emptyset$, while the assertion that B has no limit points in A is equivalent to $A \cap \overline{B} = \emptyset$. Thus, we are lead to conclude that E is disconnected relative to itself if and only if $A \cap \overline{B} = \overline{A} \cap B = \emptyset$. E is disconnected relative to itself, however, if and only if E is disconnected in M (Why?). This proves the desired result.

3. Let $\mathcal{U}, V$ be open, disjoint subsets of M such that $E \cup F \subset \mathcal{U} \cup V$. Let $x \in E \cap F \neq \emptyset$. Then, without loss of

generality, $x \in \mathcal{U}$. Since E is connected, $E \subset \mathcal{U}$. Similarly, $F \subset \mathcal{U}$. (why?) It follows that $E \cup F \subset \mathcal{U}$.

We have shown that $E \cup F$ does not have a disconnection. Hence $E \cup F$ is connected as desired.

4. Suppose that M is disconnected with disconnection $\{A, B\}$. Let $x \in A$ and $y \in B$. By hypothesis, x and y are contained in some connected set $E \subset M$. But then $A \cap E \neq \emptyset$, $B \cap E \neq \emptyset$, $(A \cap E) \cap (B \cap E) = \emptyset$, and $E = (A \cap E) \cup (B \cap E)$. Hence $\{(A \cap E), (B \cap E)\}$ is a disconnection of E, which is a contradiction.

5. Suppose that $\bar{E} \cap \bar{F} = \emptyset$. Define $f : (M, d) \mapsto \mathbb{R}$ by
$$f(x) = \frac{d(x, \bar{E})}{d(x, \bar{E}) + d(x, \bar{F})}.$$
Then f is continuous (why?) and has the property $f^{-1}(\{0\}) = \bar{E}$ and $f^{-1}(\{1\}) = \bar{F}$. Thus the sets $f^{-1}(-\frac{1}{3}, \frac{1}{3})$, $f^{-1}(\frac{2}{3}, \frac{4}{3})$ are open, nonempty, disjoint sets in (M, d) that contain $\bar{E}$ and $\bar{F}$ respectively. But then $E \cup F \subset \bar{E} \cup \bar{F} \subset f^{-1}(-\frac{1}{3}, \frac{1}{3}) \cup f^{-1}(\frac{2}{3}, \frac{4}{3})$, which shows that $E \cup F$ is disconnected. Hence $E \cup F$ connected $\Rightarrow \bar{E} \cap \bar{F} \neq \emptyset$.

6. Let A be connected and suppose that $A \subset B \subset \bar{A}$. Recall that $\bar{A} = A \cup A'$, where A' is the set of limit points of A. Thus $B \setminus A \subset A'$. Let $\mathcal{U}$ and V be open, disjoint sets such that

$B \subset \mathcal{U} \cup V$. This, of course, means that $A \subset \mathcal{U} \cup V$ so, without loss of generality, $A \subset \mathcal{U}$ (why?). We wish to show that $B \subset \mathcal{U}$ as well.

Let $x \in B \setminus A$, if $x \in V$ then there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset V$. But x is a limit point of $A \subset \mathcal{U}$ so there must be some $y \in A \cap B_\epsilon(x) \subset V$. This contradicts the assumption that $\mathcal{U} \cap V = \emptyset$. Hence $x \notin V$ or, equivalently, $x \in \mathcal{U}$. Thus $B \setminus A \subset \mathcal{U}$ so $B = A \cup B \setminus A \subset \mathcal{U}$, which proves that B is connected.

7. False. Let $A = [0,1]$, $B = [0,1] \cup [2,3]$, $C = [0,3]$. Then A and C are connected with $A \subset B \subset C \subset \mathbb{R}$, but B is disconnected.

8. Let (M,d) be connected and suppose $a, b \in M$ with $a \neq b$. Define $f: M \mapsto \mathbb{R}$ by $f(x) = d(x,a)$. Then f is continuous (why?) and $f(a) = 0$, while $f(b) = d(b,a) > 0$. Since M is connected, it follows that $(f(a), f(b)) \subset f(M)$. In particular, $f(M)$ is uncountable. This in turn implies that M is also uncountable. (Why?)

9. Let $f: [a,b] \mapsto [a,b]$ be continuous. Define $g: [a,b] \mapsto \mathbb{R}$ by $g(x) = f(x) - x$. Then g is continuous. Moreover, since $a \leq f(x) \leq b$, $f(x) - a \geq 0$ and $f(x) - b \leq 0$ for all $x \in [a,b]$. Thus $g(a) = f(a) - a \geq 0$ and $g(b) = f(b) - b \leq 0$. It follows that there is $x \in [a,b]$, $g(x) = 0$ (why?). This x satisfies $f(x) = x$.

10. Let $f: [0,2] \mapsto \mathbb{R}$ be continuous with $f(0) = f(2)$. Define $g: [0,2] \mapsto \mathbb{R}$ by $g(x) = f(x+1) - f(x)$. Then g is continuous and

$g(0) = f(1) - f(0)$, while $g(1) = f(2) - f(1) = f(0) - f(1) = -(f(1) - f(0))$.
Thus, for $r = |f(1) - f(0)|$, $0 \in (-r, r) \subset g([0,2])$. In particular, $0 = g(x)$ for some $x \in [0,2]$, which implies that $f(x) = f(x+1)$ as desired.

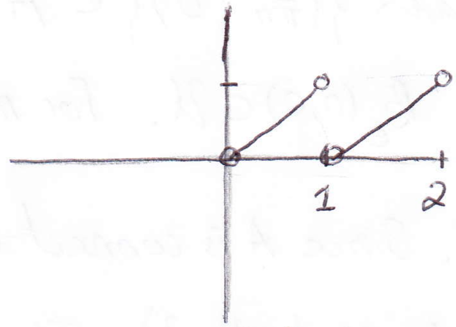
11. Suppose that $f: \mathbb{R} \mapsto \mathbb{R}$ satisfies $f(\mathbb{Q}) \subset \mathbb{R} \setminus \mathbb{Q}$ and $f(\mathbb{R} \setminus \mathbb{Q}) \subset \mathbb{Q}$. Then f is not constant (why?). We will show that f is not continuous by proving that any nonconstant function with countable range and domain $\mathbb{R}$ cannot be continuous. First, observe that $f(\mathbb{R}) = f(\mathbb{R} \setminus \mathbb{Q}) \cup f(\mathbb{Q}) \subset \mathbb{Q} \cup f(\mathbb{Q})$ is countable. If f is not constant and continuous, let $a, b \in \mathbb{R}$ such that $f(a) < f(b)$. Then $(f(a), f(b)) \subset f(\mathbb{R})$, which would contradicts our earlier assertion that $f(\mathbb{R})$ is countable.

12. Suppose A and B are closed subsets of M such that $A \cap B$ is connected. If $A \cap B = \emptyset$ then $\{A, B\}$ is a disconnection of $A \cup B$. That is, A and B are clopen subsets of $A \cup B$ (relative to $A \cup B$). If $A \cap B \neq \emptyset$ and A is disconnected, let $\{E, F\}$ be a disconnection of A (relative to A). Then E and F are clopen subsets of A. Since A is closed in M, E and F are actually closed subsets of M (why?). Thus, $A = E \cup F$. Furthermore, since $A \cap B \subset A$ is connected, without loss of generality, $A \cap B \subset E$. We can write $A \cup B = E \cup F \cup B$, where $B \cap F \subset B \cap A \subset E$, implying that $B \cap F = \emptyset$. Thus $A \cup B = (E \cup B) \cup F$ where $\{(E \cup B), F\}$ is a disconnection of $A \cup B$. In particular, A must be connected given that $A \cup B$ is connected.

13. (a) Let $f : (0,1) \cup (1,2) \mapsto \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & \text{if } x \in (0,1) \\ x-1 & \text{if } x \in (1,2) \end{cases}$$

Then f is a continuous function with disconnected domain.



Graph of f

However, $f\big((0,1) \cup (1,2)\big) = (0,1)$ - a connected set.

(b) Let $f : D \mapsto \mathbb{R}$ be continuous. Suppose that the set $G = \{(x, f(x)) : x \in D\}$ is a connected subset of $\mathbb{R}^2$. Define $\pi_1 : \mathbb{R}^2 \mapsto \mathbb{R}$ by $\pi_1(x,y) = x$. Then π_1 is continuous and $\pi_1(G) =$
$= D$ is connected.

14. Let $f : [0,1] \mapsto \mathbb{R}$ be given by

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$$

Set $A = \{(x, \sin(1/x)) : x \in (0,1]\}$ and $B = \{(0,0)\}$. Then the graph of f is $G = A \cup B$. Clearly f is not continuous at $x = 0$, but notice that $f|_{(0,1]}$ is continuous. In particular, A is connected (can you justify this claim?). Since B contains only one point, B is connected as well. To see that $A \cup B$ is connected, let $\mathcal{U}$ and V be open, disjoint subsets of $\mathbb{R}^2$ such that $A \cup B \subset \mathcal{U} \cup V$. Suppose that $B \subset \mathcal{U}$, we wish to prove that $A \subset \mathcal{U}$. To do this, observe that $\{(\frac{1}{\pi n}, 0)\} \subset A$ that satisfies $\lim_{n\to\infty} (\frac{1}{\pi n}, 0) = (0,0)$. Let $B_\epsilon(0,0) \subset \mathcal{U}$. For $n \geq N$, $(\frac{1}{\pi n}, 0) \in B_\epsilon(0,0)$. Hence $A \cap \mathcal{U} \neq \emptyset$. Since A is connected and $A \subset \mathcal{U} \cup V$, this implies that $A \subset \mathcal{U}$ (why?). Thus we have shown that $A \cup B$ does not have a disconnection. Hence $A \cup B$ is connected as desired. Notice that $A \cup B$ is not pathwise connected, since $A \cap B = \emptyset$.