Solutions to review questions For exam 1

(1.1)

1. A vector is a displacement between point a (where the particle began its journy) and point b (where the particle ended its journy). A point is simply the particle's position in space.

2. Instead of describing a curve or a surface in terms of the coordinate axes, the curve or surface may sometimes be more simply described with a different set of variables. For example $P = \{(s+2t-1, t, 4s+2) : s,t \in \mathbb{R}\}$ is a plane spanned by the vectors $(1,0,4)$ and $(2,1,0)$. This plane looks just like the xy plane to any pedestrian walking on it. However, the positions on the plane relative to the xyz coordinate system are given by $(x,y,z) = (x(s,t), y(s,t), z(s,t))$ where $x(s,t) = s+2t-1$, $y(s,t) = t$, and $z(s,t) = 4s+2$.

3. $L(t) = (1,0,-3) + t(-3,-10,1) = (1-3t, -10t, -3+t)$.

This line is the $\underline{\text{Set}}$ $L = \{(x,y,z) : (x,y,z) = (1-3t, -10t, -3+t), t \in \mathbb{R}\}$

$L(t)$ represents the position at time t of the particle that is tracing this line.

4. To be parallel, this line must have direction vector that is a non-zero scalar multiple of the direction vector of $L(t)$ (Ex. 3). Thus, we may take $-3i-10j+k = (-3,-10,1)$ to be the direction vector.

The desired line is parametrized by $S(t) = (1,1,1) + t(-3,-10,1)$

5. Two lines are perpendicular iff their directional vectors are perpendicular. There are infinitely many vectors that are perpendicular to $(-3,-10,1)$ and infinitely many lines through $(1,0,-3)$ that are perpendicular to the line from (Ex. 3). Take $\vec{n} = (0,1,10)$. Then $\vec{n}\cdot(-3,-10,1) = (0,1,10)\cdot(-3,-10,1) = 0$ Hence $\vec{n} \perp (-3,-10,1)$. It follows that the line through $(1,0,-3)$ in the direction $\vec{n}$ is perpendicular to the line from (Ex. 3):
$p(t) = (1,0,-3) + t(0,1,10)$ is a parametrization of this line.
(Note that the solution is not unique. It would have been unique if we asked to find a perpendicular line through $(1,0,-3)$ that <u>intersects</u> the line in (Ex. 3)).

6. $(1,0,-3) + t\big((6,7,6)-(1,0,-3)\big) = (1,0,-3) + t(5,7,9) = L(t)$
Note that $L(0) = (1,0,-3)$ and $L(1) = (6,7,6)$. Thus the particle that parametrizes this line will be observed in position $(1,0,-3)$ at present time. In 1 time unit, the particle will have traveled to position $(6,7,6)$.
What is a parametrization $p(t)$ s.t. $p(0) = (6,7,6)$ and $p(1) = (1,0,-3)$? $P(t)$ is another solution.

7. $L(t)$ has direction $(-1,1,1)$ and $p(t)$ has direction $(3,1,2)$. Since $(-1,1,1)\cdot(3,1,2) = -3+1+2=0$, the lines are perpendicular to each other.

To see whether or not the lines intersect, set $L(t)=p(s)$:

$(-2-t,\ 3+t,\ -2+t) = (-1+3s,\ 2+s,\ -3+2s) \Rightarrow$

$$\begin{cases} -2-t = -1+3s \\ 3+t = 2+s \\ -2+t = -3+2s \end{cases}$$

The last equation implies that $t=-1+2s$. Substituting in the middle equation, we get $3+(-1+2s) = 2+s$ or $2s=s \Rightarrow s=0 \Rightarrow t=-1$. Checking:

$$-2-(-1) \stackrel{?}{=} -1+3\cdot 0 \ \checkmark$$
$$3-1 \stackrel{?}{=} 2+0 \ \checkmark$$
$$-2-1 \stackrel{?}{=} -3+2\cdot 0 \ \checkmark$$

Thus the lines traced by $p(t)$ and $L(t)$ intersect but the particles that traced them <u>do not</u> collide (why?).

8. $P = \{(0,1,-6)+s(2,1,-4)+t(-3,-5,2) : s,t\in\mathbb{R}\} =$

$= \{(2s-3t,\ 1+s-5t,\ -6-4s+2t) : s,t\in\mathbb{R}\}$

9. S is spanned (generated) by the vectors $(2,-1,7)$ and $(-6,3,-21) = -3(2,-1,7)$. Since the latter is a scaler multiple of the first vector, the two vectors are linearly dependent.

Consequently S is really a line.

Alternative solution:

let $u = s-3t$ then $(2s-6t+1, 3t-s, 7s-21t+4) =$

$\Rightarrow (2(s-3t)+1, -(s-3t), 7(s-3t)+4) = (2u+1, -u, 7u+4) =$

$= (1,0,4) + u(2,-1,7) = L(u)$ note that $L(u)$ is a parametrization of a line through $(1,0,4)$ with direction vector $(2,-1,7)$.

10. $S = \{(-1,-4,7) + r(1,0,1) + s(2,2,0) + t(0,3,3)\}$

Since $r,s \in [0,1]$ and $t \in [0,2]$, S is the parallelopiped spanned by the vectors $(1,0,1)$, $(2,2,0)$, and $2(0,3,3) = (0,6,6)$ with corner at $(-1,-4,7)$.

(1.2)

1. Review the notes. Ask yourself "what is proportionality?"

2. $(2,1,-4)\cdot(-3,-5,2) = -6-5-8 = -19$

3. $\theta = \cos^{-1}\left(\frac{-19}{\|(2,1,-4)\|\,\|(-3,-5,2)\|}\right) = \cos^{-1}\left(\frac{-19}{\sqrt{21}\sqrt{38}}\right)$

4. $\frac{(2,1,-4)}{\|(2,1,-4)\|} = \frac{1}{\sqrt{21}}(2,1,-4) \equiv$ unit vector in the direction of $v = (2,1,-4)$; $\frac{-1}{\sqrt{21}}(2,1,-4) \equiv$ unit vector in the opposite direction

5. $P_w(v) = P_{(-3,-5,2)}(2,1,-4) = \frac{(2,1,-4)\cdot(-3,-5,2)}{\|(-3,-5,2)\|^2}(-3,-5,2) =$

$= \frac{-19}{38}(-3,-5,2)$

6. Two vectors are linearly dependent if they generate the same line. Mathematically $\vec{v}, \vec{w}$ are linearly dependent if there exist scalars $a, b \in \mathbb{R}$ that are <u>not both</u> 0 s.t. $a\vec{v} + b\vec{w} = \vec{0}$.

Three vectors $\vec{v}, \vec{w}, \vec{u}$ are linearly dependent if they all lie on the same plane. Mathematically, if there are scalars $a, b, c \in \mathbb{R}$, <u>not all</u> 0 s.t. $a\vec{v} + b\vec{w} + c\vec{u} = \vec{0}$.

7. Review lecture notes for the proof. Triangle inequality states, in geometric terms, that the shortest distance between any two points in Euclidian space is the length of the line segment connecting them.

(1.3)

1. Review lecture notes.

2. $$\begin{vmatrix} 2 & 2 & 0 \\ -3 & 0 & -3 \\ 0 & 5 & 5 \end{vmatrix} = 2\begin{vmatrix} 0 & -3 \\ 5 & 5 \end{vmatrix} - 2\begin{vmatrix} -3 & -3 \\ 0 & 5 \end{vmatrix} + 0\begin{vmatrix} -3 & 0 \\ 0 & 5 \end{vmatrix} =$$

$= 2 \cdot 15 + 2 \cdot 15 = 4 \cdot 15 = 60$ since the volume is just the absolute value of the determinant, the answer is 60.

3. $$\begin{vmatrix} n & n+1 & n+2 \\ n+3 & n+4 & n+5 \\ n+6 & n+7 & n+8 \end{vmatrix} = \begin{vmatrix} n & n+1 & n+2 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} = 0$$

4. $$\begin{vmatrix} 3a_{21} & 3a_{22} & 3a_{23} \\ a_{11} & a_{12} & a_{13} \\ b_{31}-a_{31} & b_{32}-a_{32} & b_{33}-a_{33} \end{vmatrix} = \begin{vmatrix} 3a_{21} & 3a_{22} & 3a_{23} \\ a_{11} & a_{12} & a_{13} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} -$$

$$- \begin{vmatrix} 3a_{21} & 3a_{22} & 3a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} +$$

$$+ 3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 3\big(-1 + (-4)\big) = 3(-5) = -15$$

5. the triangle has "edge vectors" $\vec{v} = (3,3)-(1,1) = (2,2)$ and $\vec{w} = (2,6)-(1,1) = (1,5)$. The area of the triangle is half the area of the parallelogram with corresponding edges:

$$\frac{1}{2}\left|\det\begin{pmatrix} 2 & 2 \\ 1 & 5 \end{pmatrix}\right| = \frac{1}{2}\left|2\det\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}\right| = \left|\begin{matrix} 1 & 1 \\ 1 & 5 \end{matrix}\right| = 5-1 = 4$$

6. Review lecture notes.

7. $$\begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ -3 & 0 & -3 \end{vmatrix} = -6i + 6j + 6k = (-6, 6, 6)$$

$$-2b \times a = -2(b \times a) = -2(-1)(a \times b) = 2(a \times b) = (-12, 12, 12)$$

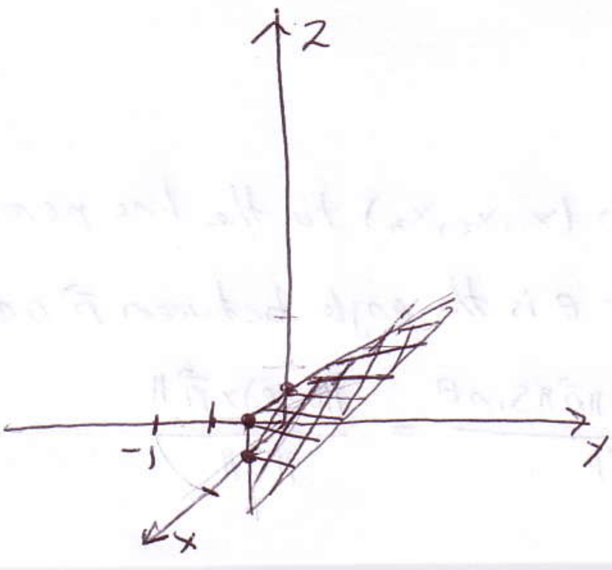
8. Let $\vec{v} = (-1,0,1) - (1,1,1) = (-2,-1,0)$ and $\vec{w} = (0,2,3) -$
$- (1,1,1) = (-1,1,2)$

Define $\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ -2 & -1 & 0 \\ -1 & 1 & 2 \end{vmatrix} = -2i + 4j - 3k = (-2, 4, -3)$

The desired plane satisfies the equation $\vec{n} \cdot (x-1, y-1, z-1) = 0$
or $(-2,4,-3) \cdot (x-1, y-1, z-1) = -2x + 2 + 4y - 4 - 3z + 3 =$
$= -2x + 4y - 3z + 1 = 0$

Remark: any $\lambda \neq 0$ allows the equation of the plane to be written as $-2\lambda x + 4\lambda y - 3\lambda z + \lambda = 0$.

9. The plane's intersection with the xy plane is the line $-2x + 4y + 1 = 0$ when $y = 0$ we see that $(\frac{1}{2}, 0, 0)$ is on the plane. Also, when $x=0$ we see that $(0, -\frac{1}{4}, 0)$ is on the plane,
The plane's intersection with the xz plane is the line $-2x - 3z + 1 = 0$
when $x=0$; we see that $(0, 0, \frac{1}{3})$ is on the plane. These 3 points are sufficient to graph it.



10. The distance from a point (x_1, y_1, z_1) to the plane with equation $Ax+By+Cz+D=0$ is $\dfrac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$

Hence the desired distance is $\dfrac{|3\cdot 6-5(-10)+2(3)+2|}{\sqrt{3^2+5^2+2^2}} =$

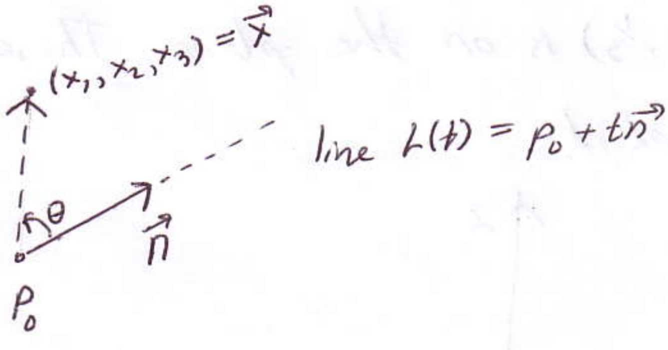
$$= \frac{|18+50+6+2|}{\sqrt{38}} = \frac{76}{\sqrt{38}}$$

11. $\|v\times w\| = \|v\|\,\|w\|\sin\theta$ where θ is the angle from v to w.

12. The area of the parallelogram is the norm of the cross product of the vectors that form its sides.

$$\begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = i-2j+2k = (1,-2,2).$$ The area of given parallelogram is $\|(1,-2,2)\| = \sqrt{1+4+4} = \sqrt{9} = 3$

13.



The distance from (x_1,x_2,x_3) to the line parametrized by $L(t)=p_0+t\vec{n}$ is $\|\vec{x}-p_0\|\sin\theta$ where θ is the angle between $\vec{n}$ and $\vec{x}-p_0$ (why?).

$$\|\vec{x}-p_0\|\sin\theta = \frac{\|\vec{x}-p_0\|\,\|\vec{n}\|\sin\theta}{\|\vec{n}\|} = \frac{\|(x-p_0)\times\vec{n}\|}{\|\vec{n}\|}$$

In our case $\vec{x}=(1,2,3)$, $p_0=(0,1,0)$ and $\vec{n}=(1,2,1)$.
Therefore $\vec{x}-p_0=(1,1,3)$ and $(\vec{x}-p_0)\times\vec{n}=\begin{vmatrix} i & j & k \\ 1 & 1 & 3 \\ 1 & 2 & 1 \end{vmatrix}=$
$=-5i+2j+k=(-5,2,1)$

Therefore the distance is $\frac{\|(-5,2,1)\|}{\|(1,2,1)\|}=\frac{\sqrt{25+4+1}}{\sqrt{1+4+1}}=\frac{\sqrt{30}}{\sqrt{6}}$
$=\sqrt{\frac{30}{6}}=\sqrt{5}$.

(1.5)

1. $e_3=(0,0,1,0,0)$
2. $a\cdot b=1\cdot3+0\cdot3+4(-1)+(-2)6=3-4-12=-13.$
3. $AB=\begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 6 & 19 \end{pmatrix}$ BA is undefined.

$3B=\begin{pmatrix} 6 & 15 \\ 0 & 12 \\ 0 & 3 \\ 0 & 3 \end{pmatrix}$ A-B is undefined

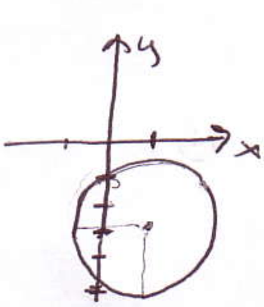
$A^T=\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 4 \end{pmatrix}$

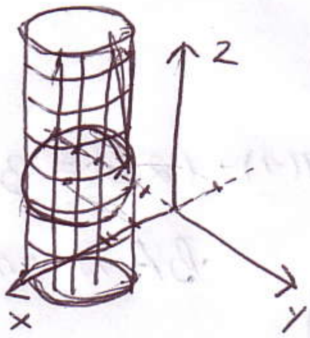
4. $-2C=\begin{pmatrix} 0 & 2 & 0 & 2 \\ -8 & -16 & 0 & -2 \\ -2 & 0 & 0 & -18 \end{pmatrix}$ so $A-2C=\begin{pmatrix} 1 & 1 & 1 & 1 \\ -6 & -18 & 0 & -1 \\ 1 & 0 & 0 & -14 \end{pmatrix}$

$$A+C = \begin{pmatrix} 1 & -2 & 1 & -2 \\ 6 & 6 & 0 & 2 \\ 4 & 0 & 0 & 13 \end{pmatrix}$$

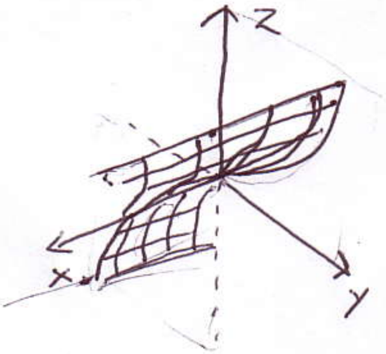
(2.1)

1. In $\mathbb{R}^2$ this equation describes a circle with center $c=(1,-3)$ and radius $r=2$. In $\mathbb{R}^3$ this is a cylinder generated by projecting this circle along the z-axis.





2. $f(x,y)=y^3$ is a cylinder obtained by projecting the graph of $z=y^3$ in the yz plane along the x-axis



3. $f(x,y) = x^2+2x-4x+y^2-7 = x^2-2x+1+y^2-8 = (x-1)^2+y^2-8$

Its graph is a paraboloid shifted so that the lowest point of the "bowl" is at $(1,0,-8)$

4. $f(x,y) = \frac{x^2}{9} + \frac{y^2}{25} = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2$ Therefore its graph is an elliptic parabuloid obtained by stretching a circular parabuloid by a factor of 3 on the x-axis and by a factor of 5 on the y-axis.

circuler parabuloid

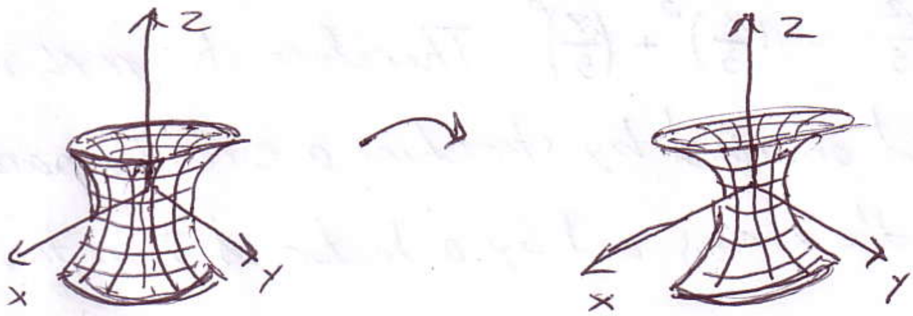
eliptic parabuloid

5. This equation can be written as $z^2 = \frac{x^2}{9} + \frac{y^2}{25} \Rightarrow$ $z = \pm\sqrt{\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2}$ which is an eliptic "hour-glass" cone:

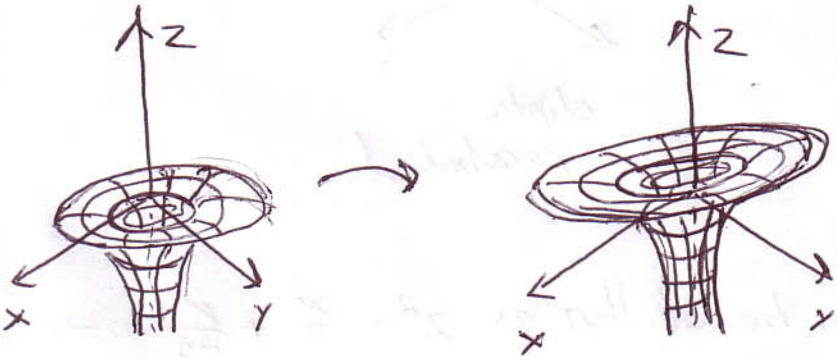
circuler "hour-glass"

eliptic "hour-glass".

6 This equation can be written as $z^2 = \frac{x^2}{9} + \frac{y^2}{25} - 1 \Rightarrow$ $z = \pm\sqrt{\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 - 1}$ which is a "circle-into-elipse" stretch of $\omega = \pm\sqrt{u^2+v^2-1}$. The latter curve is a revolution of $\omega = \pm\sqrt{u^2-1}$ about the ω-axis. The corresponding graph to $\omega = \pm\sqrt{u^2+v^2-1}$ is therefore the circuler hyperbuloid that is deformed into an eliptic hyperbuloid.



7. $f(x,y) = \ln\left(\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2\right)$;



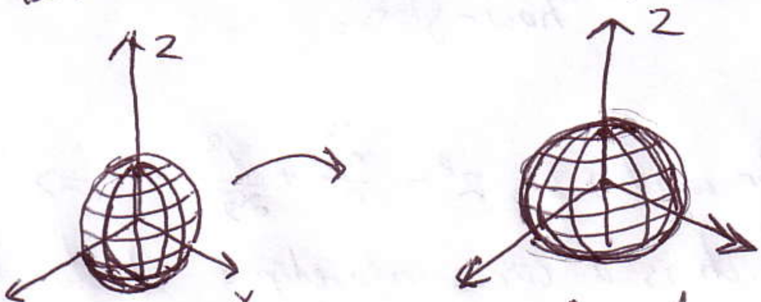
circuler worm-hole

eliptic worm-hole.

8. $(x-3)^2 + (y+2)^2 + (z-1)^2 = 25$

9. $z^2 = 1 - x^2 - \frac{y^2}{4} \Rightarrow z = \pm\sqrt{1-(x^2+(\frac{y}{2})^2)}$

or $x^2 + (\frac{y}{2})^2 + z^2 = 1$ which is an ellipsoid ;



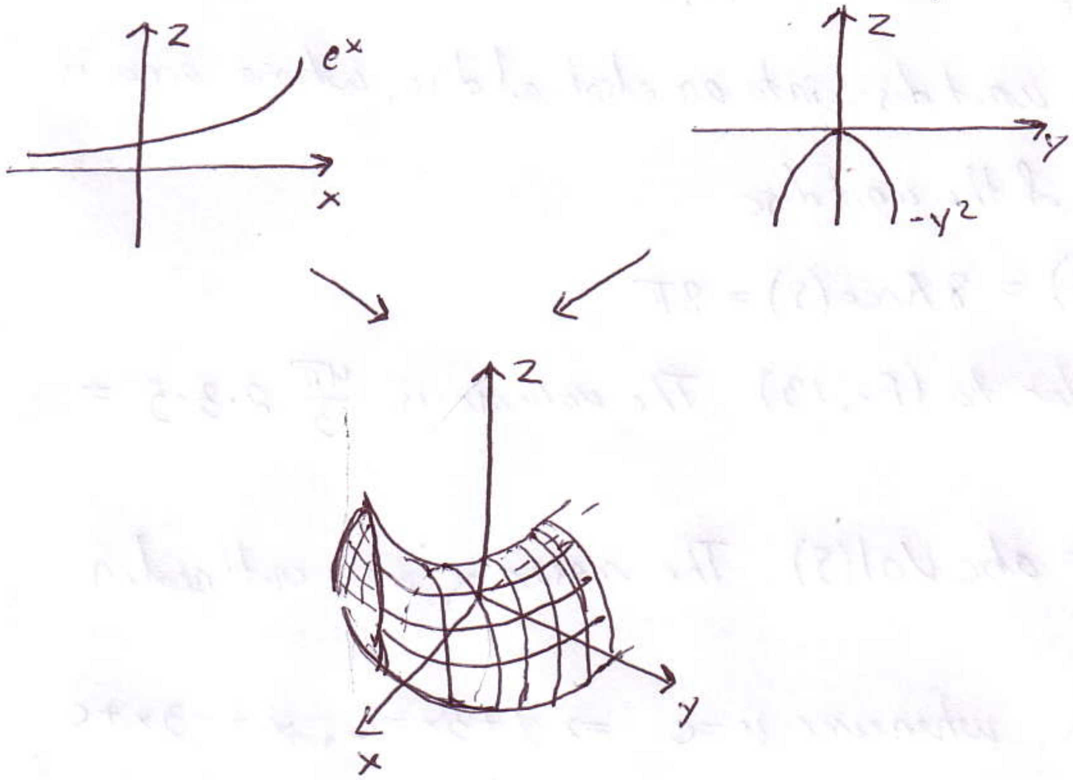
sphere

ellipsoid.

10. $z^4 = (x^2+y^2)^2 \Rightarrow z^2 = x^2+y^2 \Rightarrow z = \pm\sqrt{x^2+y^2}$

which is a circular "hour-glass" cone.

11. $f(x,y) = e^x - y^2$ is of the form $g(x)+h(y)$ so we can use the "fish and the wire" method to visualize this graph



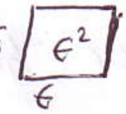
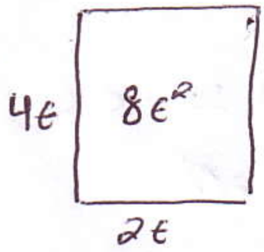
12. $f(x,y) = \begin{vmatrix} x & y \\ y & x \end{vmatrix} = x^2 - y^2$. We can visualize this graph by the "fish and wire" method or by describing it in cylindrical coordinates: $f(r\cos\theta, r\sin\theta) = r^2\cos 2\theta$,

13. Let $S = \{(x,y) : x^2+y^2 \le 1\}$ then S is a circular disc.

Let $T(x,y) = (2x, 4y)$ then $T(S) =$

$= \{(2x, 16y) : x^2+y^2 \le 1\} = \{(u,v) : (\frac{u}{2})^2 + (\frac{v}{4})^2 \le 1\} = \{(x,y) : \frac{x^2}{4} + \frac{y^2}{16} \le$

$= \{(x,y) : \frac{x^2}{2} + \frac{y^2}{8} \le 2\}$ which is the eliptical disc of semi-major radius 4 and semiminor radius 2.

Since T modifies the square of area ϵ^2  into a rectangle  with area $8\epsilon^2$ (8 times the area of the square),

T stretches the unit disc into an eliptical disc, whose area is 8 times the area of the unit disc.

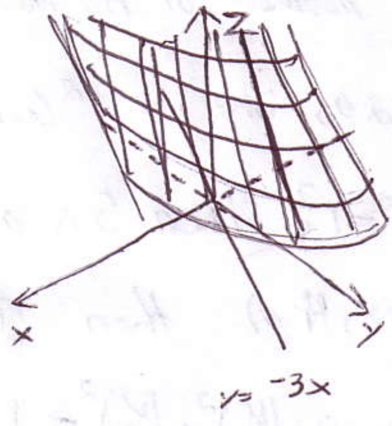
Hence $\text{Area}(T(S)) = 8\,\text{Area}(S) = 8\pi$

14. This is similar to (Ex, 13). The volume is $\frac{4\pi}{3}\, 2\cdot 3\cdot 5 =$
$= 40\pi$.

15. $\text{Vol}(T(S)) = abc\,\text{Vol}(S)$ The reasoning was outlined in (Ex, 13).

16. let $u = y+3x$. whenever $u = c \Rightarrow y+3x = c$, $y = -3x+c$

Thus $f(x,y) = e^{y+3x}$ is a cylinder with respect to the coordinate system in which the x-axis is replaced by the line $y=-3x$



Linear Transformations

1. A linear map satisfies $T(x+y) = T(x) + T(y)$ & for any scalar $a \in \mathbb{R}$, $T(ax) = aT(x)$.

Linear maps are generalizations of functions of the form $f(x) = m$

2. $TS(x,y) = T(-y, x-y, 2y) = (x, -2y, 3x-5y)$

ST is undefined, because $T(x,y,z)$ is a vector in $\mathbb{R}^3$ and is therefore not an element in the domain of S.

3. $6T(x,y,z) = (6x+6y+6z, 12x, 18y-6z)$

$(T-2U)(x,y,z) = (x+y+z, 2x, 3y-z) - (2x, 2x+2y, 2x+2y-2$

$= (-x+y+z, -2y, -2x+y+z)$

$UT(x,y,z) = U(x+y+z, 2x, 3y-z) = (x+y+z, 3x+y+z, 3x-2y+2$

4. $\mathcal{M}(T) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & -1 \end{pmatrix}$ $\mathcal{M}(S) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$

$\mathcal{M}(6T) = 6\mathcal{M}(T) = 6 \cdot \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 6 & 6 \\ 12 & 0 & 0 \\ 0 & 18 & -6 \end{pmatrix}$

$\mathcal{M}(T-2U) = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$ $\mathcal{M}(UT) = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & -2 & 2 \end{pmatrix} = \mathcal{M}(U)\mathcal{M}(T)$

5. $(x,y) = T^{-1}(T(x,y)) = T^{-1}(-y,x)$

Thus $T^{-1}(u,v) = (v,-u)$ or $T^{-1}(x,y) = (y,-x)$

6. let $A = \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}$, $A^{-1} = \frac{1}{7-6}\begin{pmatrix} 1 & -2 \\ -3 & 7 \end{pmatrix}^T = \begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix}$

$B = \begin{pmatrix} -1 & 3 \\ 2 & 3 \end{pmatrix}$, $B^{-1} = \frac{1}{-3-6}\begin{pmatrix} 3 & -2 \\ -3 & -1 \end{pmatrix}^T = \frac{-1}{9}\begin{pmatrix} 3 & -3 \\ -2 & -1 \end{pmatrix}$

$C = \begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix}$, C^{-1} is undefined because $\det(C) = 0$,

7. $T(x,y) = (7x+3y, 2x+y) \equiv \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = Ax^T$

$T^{-1}(x,y) \equiv A^{-1}x^T \quad A^{-1} = \frac{1}{7-6}\begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix}$

Hence $T^{-1}(x,y) \equiv \begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-3y \\ -2x+7y \end{pmatrix}$ so

$T^{-1}(x,y) = (x-3y, -2x+7y)$

$S(x,y) = (-x+3y, 2x+3y) \equiv \begin{pmatrix} -1 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = Bx^T$

so $S^{-1}(x,y) \equiv \frac{-1}{9}\begin{pmatrix} 3 & -3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

8. The answer will be revealed to the chosen

(2.2)

1. $$\lim_{(x,y)\to(1,2)} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2} = \lim_{(x,y)\to(1,2)} \frac{x(y-2)-(y-2)}{(x-1)^2+(y-2)^2} =$$

$$= \lim_{(x,y)\to(1,2)} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2} = \lim_{(u,v)\to(0,0)} \frac{uv}{u^2+v^2} =$$

$$= \lim_{r\to 0} \frac{\frac{1}{2} r^2 \sin 2\theta}{r^2} = \lim_{r\to 0} \sin 2\theta.$$ Since this limit depends on the angle θ, the limit does not exist.

2. $$\lim_{(x,y)\to(0,0)} \frac{2x^2y-2xy^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} 2xy \frac{x-y}{x^2+y^2} =$$

$$= \lim_{r\to 0} r^2 \sin 2\theta \frac{r\cos\theta - r\sin\theta}{r^2} = \lim_{r\to 0} r(\sin 2\theta[\cos\theta - \sin\theta]) = 0$$

Thus the limit exists and is equal to 0,

3. $$0 \leq \frac{|xyz|}{x^2+y^2+z^2} \leq \frac{\left(\sqrt{x^2+y^2+z^2}\right)^3}{\left(\sqrt{x^2+y^2+z^2}\right)^2} = \sqrt{x^2+y^2+z^2}$$

Hence $$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = 0$$

(whenever $\|(x,y,z)\| < \epsilon$ so is $\left|\frac{xyz}{x^2+y^2+z^2} - 0\right|$, because

$$\left|\frac{xyz}{x^2+y^2+z^2} - 0\right| = \frac{|xyz|}{x^2+y^2+z^2} \leq \sqrt{x^2+y^2+z^2} < \epsilon,$$

4. $$\lim_{(x,y,z)\to(2,5,3)} \frac{\sin(4x^2-2y+2z)}{2x^2-y+z} = \lim_{u\to 0} \frac{\sin(2u)}{u} =$$

$$= 2\lim_{u\to 0} \frac{\sin(2u)}{2u} = 2. \text{ where } u = 2x^2-y+z.$$

5. $$\lim_{(x,y)\to(0,0)} \frac{e^{2xy}-1}{y} = \lim_{(x,y)\to(0,0)} x\,\frac{e^{2xy}-1}{xy} =$$

$$= \left(\lim_{u\to 0} \frac{e^{2u}-1}{u}\right)\left(\lim_{(x,y)\to(0,0)} x\right) = \frac{d}{du}\left(e^{2u}\right)\Big|_{u=0} \cdot 0 =$$

$$= 0$$

6. $$\lim_{(x,y)\to(0,0)} \frac{\sin(2x)-2x+y}{x^3+y}\Bigg|_{y=0} = \lim_{x\to 0} \frac{\sin(2x)-2x}{x^3}$$

$$\overset{*}{=} \lim_{x\to 0} \frac{2\cos(2x)-2}{3x^2} \overset{*}{=} \lim_{x\to 0} \frac{-4\sin(2x)}{6x} =$$

$$= -\frac{4}{3}$$

$$\lim_{(x,y)\to(0,0)} \frac{\sin(2x)-2x+y}{x^3+y}\Bigg|_{y=x^3} = \lim_{x\to 0} \frac{\sin(2x)-2x+x^3}{2x^3}$$

$$\overset{*}{=} \lim_{x\to 0} \frac{2\cos(2x)-2+3x^2}{6x^2} \overset{*}{=} \lim_{x\to 0} \frac{-4\sin(2x)+6x}{12x} =$$

$$= \lim_{x\to 0} \frac{-4}{6}\,\frac{\sin 2x}{2x} + \frac{1}{2} = -\frac{2}{3} + \frac{1}{2} = \frac{3}{6} - \frac{4}{6} = \frac{-1}{6}$$

Since $-\frac{4}{3} \neq -\frac{1}{6}$ the limit does not exist.

7. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{2x^2y\cos z}{x^2+y^2} = 0$ because

$$0 \le \left|\frac{2x^2y\cos z}{x^2+y^2}\right| \le \frac{2x^2|y|}{x^2+y^2} \le \frac{2(x^2+y^2)}{x^2+y^2}|y| =$$

$$= 2|y| \le 2\sqrt{x^2+y^2}$$

Hence $\left|\dfrac{2x^2y\cos z}{x^2+y^2}\right| \le \epsilon$ whenever $\|(x,y)\| < \frac{\epsilon}{2}$.

8. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\cos x - 1 - (x^2/2)}{x^4+y^4}\Big|_{y=x} = \lim\limits_{x\to 0} \dfrac{\cos x - 1 - (x^2/2)}{2x^4} \overset{*}{=}$

$$\overset{*}{=} \lim_{x\to 0} \frac{-\sin x - x}{8x^3} \overset{*}{=} \lim_{x\to 0} \frac{-\cos x - 1}{8\cdot 3x^2} \to \frac{-2}{0} = \infty$$

$$\lim_{(x,y)\to(0,0)} \left|\frac{\cos x - 1 - (x^2/2)}{x^4+y^4}\right| \geq \lim_{(x,y)\to(0,0)} \left|\frac{\cos x - 1 - (x^2/2)}{2(x^2+y^2)^2}\right| =$$

$$\lim_{r\to 0}\left|\frac{\cos(r\cos\theta) - 1 - \frac{r^2}{2}\cos^2\theta}{2r^4}\right| \overset{*}{=} \lim_{r\to 0}\left|\frac{-\sin(r\cos\theta)\cos\theta - r\cos^2\theta}{8r^3}\right|$$

$$\overset{*}{=} \lim_{r\to 0}\left|\frac{-\cos(r\cos\theta)\cos^2\theta - \cos^2\theta}{8\cdot 3r^2}\right| = \frac{\cos^2\theta}{8\cdot 3}\lim_{r\to 0}\left|\frac{\cos(r\cos\theta) - 1}{r^2}\right|$$

$= \infty$ Hence $\lim\limits_{(x,y)\to(0,0)} \dfrac{\cos x - 1 - (x^2/2)}{x^4+y^4} = \infty$

9. $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x-y)^2}{x^2+y^2}\Big|_{y=0} = \lim\limits_{x\to 0} \dfrac{x^2}{x^2} = 1$ while

$\lim\limits_{(x,y)\to(0,0)} \dfrac{(x-y)^2}{x^2+y^2}\Big|_{x=y} = \lim\limits_{x\to 0} \dfrac{0}{2x^2} = 0$ so lim does not exist.

10. $\lim\limits_{(x,y,z)\to(2,5,1)} 4x-3y+z = 4\cdot 2-3\cdot 5+1=-6$

To prove this, note that $|4x-3y+z-(4\cdot 2-3\cdot 5+1)| =$

$= |4(x-2)-3(y-5)+1(z-1)| \leq \|(4,-3,1)\|\,\|(x-2,y-5,z-1)\|$

Hence $|4x-3y+z+6|<\epsilon$ whenever $\|(x-2,y-5,z-1)\|<\dfrac{\epsilon}{\|(4,-3,1)\|}$

so set $\delta(\epsilon)=\dfrac{\epsilon}{\|(4,-3,1)\|}$

11. The same line of reasoning as in (Ex 10) works.

12. $\|(7x+2y+8z-14,\ x-y-z+1)\| = \|(7(x-0)+2(y+1)+8(z-2),\ 1(x-0)-1(y+1)-($

$\leq |7(x-0)+2(y+1)+8(z-2)| + |1(x-0)-1(y+1)-(z-2)| \leq$

$\leq \|(7,2,8)\|\,\|(x,y+1,z-2)\| + \|(1,-1,-1)\|\,\|(x,y+1,z-2)\| =$

$= \left(\|(7,2,8)\|+\|(1,-1,-1)\|\right)\|(x,y+1,z-2)\| < \epsilon$

so you could set $\delta(\epsilon)=\dfrac{\epsilon}{\|(7,2,8)\|+\|(1,-1,-1)\|}$