Jointly Distributed Random Variables Lecture 1

Joint Distribution Functions

$$F(a,b) = P(X \leq a, Y \leq b) \quad -\infty < a, b < \infty$$

is the joint cumulative distribution function of X and Y.

The Distribution of X is

$$F_X(a) = P(X \leq a) = P(X \leq a, Y < \infty)$$

$$= P\left(\lim_{b \to \infty} \{X \leq a, Y \leq b\}\right) = \lim_{b \to \infty} F(a,b)$$

$$= F(a, \infty)$$

Similarly $F_Y(b) = F(\infty, b)$.

In theory, all joint probability statements about X and Y can be made in terms of their joint distribution.

Ex. $P(X > a, Y > b) = P(X < \infty, Y > b) -$

$- P(X \leq a, Y > b) = P(X < \infty, Y < \infty) - P(X < \infty, Y \leq b)$

$- P(X \leq a, Y < \infty) + P(X \leq a, Y \leq b)$

$= 1 - F(\infty, b) - F(a, \infty) + F(a,b).$

<u>Ex.</u> $P(a_1 < X \leq a_2,\ b_1 < Y \leq b_2) =$

$= \underline{P(X \leq a_2,\ b_1 < Y \leq b_2)} - \underline{P(X \leq a_1,\ b_1 < Y \leq b_2)}$

$= \underline{P(X \leq a_2, Y \leq b_2)} - \underline{P(X \leq a_2, Y \leq b_1)}$

$= - \underline{P(X_1 \leq a_1, Y \leq b_2)} + \underline{P(X \leq a_1, Y \leq b_1)}$

$= F(a_2, b_2) - F(a_2, b_1) - F(a_1, b_2) + F(a_1, b_1)$

If X, Y are discrete random variables

$p(x,y) = P(X = x, Y = y)$ is called the probability mass function.

$$P_X(x) = \sum_{Y:\, y \in \mathbb{R}} p(x,y) \quad ; \quad P_Y(y) = \sum_{X:\, x \in \mathbb{R}} p(x,y)$$

<u>Ex.</u> 3 balls sampled randomly from urn containing 3 red, 4 white, and 5 blue balls. If $X = \#$ of red balls in sample and $Y = \#$ of white, the joint

probability mass function is given by

$$p(0,0) = \frac{\binom{5}{3}}{\binom{12}{3}} = \frac{10}{220} \qquad p(0,1) = \frac{\binom{5}{2}\binom{4}{1}}{\binom{12}{3}} = \frac{40}{220}$$

$$p(0,2) = \frac{\binom{5}{1}\binom{4}{2}}{\binom{12}{3}} = \frac{30}{220} \qquad p(0,3) = \frac{\binom{4}{3}}{\binom{12}{3}} = \frac{4}{220}$$

$$p(1,0) = \frac{\binom{5}{2}\binom{3}{1}}{\binom{12}{3}} = \frac{30}{220} \qquad p(1,1) = \frac{\binom{5}{1}\binom{3}{1}\binom{4}{1}}{\binom{12}{3}} = \frac{60}{220}$$

$$p(1,2) = \frac{\binom{3}{1}\binom{4}{2}}{\binom{12}{3}} = \frac{18}{220} \qquad p(2,0) = \frac{\binom{5}{1}\binom{3}{2}}{\binom{12}{3}} = \frac{15}{220}$$

$$p(2,1) = \frac{\binom{3}{2}\binom{4}{1}}{\binom{12}{3}} = \frac{12}{220} \qquad p(3,0) = \frac{\binom{3}{3}}{\binom{12}{3}} = \frac{1}{220}$$

$i \backslash j$	0	1	2	3	Row sum $= P(X=i)$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
Column sum	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	

Jointly Continuous Variables

X and Y are jointly continuous if there is a continuous function $f(x,y)$ such that

$$P\big((X,Y) \in C\big) = \iint_{(x,y)\in C} f(x,y)\,dx\,dy .$$

The function $f(x,y)$ is called the joint probability density function (jpdf).

The cumulative distribution is given by

$$F(a,b) = P(X \leq a, Y \leq b) = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy$$

and the density function may be obtained as follows:

$$\frac{\partial}{\partial a}\frac{\partial}{\partial b} F(a,b) = \frac{\partial^2 F}{\partial a\,\partial b}(a,b) = \frac{\partial}{\partial a}\int_{-\infty}^{a} f(x,b)\,dx$$

$$= f(a,b).$$

Observe that $P(X \in A) = P(X \in A, -\infty < Y < \infty) =$

$$= \int_A \int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_A f_X(x)\,dx$$

where $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

Similarly $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

<u>Ex.</u> $f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0<x<\infty, \quad 0<y<\infty \\ 0 & \text{otherwise} \end{cases}$

Compute

(a) $P(X>1, Y<1)$ (b) $P(X<Y)$

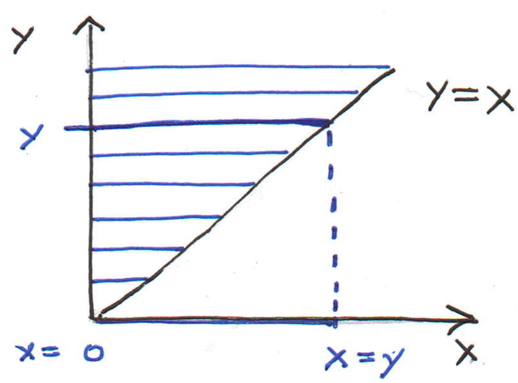
(c) $P(X<a)$

<u>Solution:</u>

(a) $P(X>1, Y<1) = \int_1^{\infty}\int_0^1 2e^{-x}e^{-2y}\,dy\,dx$

$= \left(\int_1^{\infty} e^{-x}\,dx\right)\left(\int_0^1 2e^{-2y}\,dy\right) = e^{-1}(1-e^{-2})$

(b) $P(X<Y) = \int_0^{\infty}\int_0^{y} 2e^{-x}e^{-2y}\,dx\,dy$



$$\int_0^\infty 2e^{-2y}\int_0^y e^{-x}\,dx\,dy = \int_0^\infty 2e^{-2y}(1-e^{-y})\,dy =$$

$$= -e^{-2y} + \frac{2}{3}e^{-3y}\Big|_0^\infty = 1-\frac{2}{3} = \frac{1}{3}.$$

(c) $P(X<a) = \int_{-\infty}^{a}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx =$

$$= \int_0^a\int_0^\infty 2e^{-x}e^{-2y}\,dy\,dx = \left(\int_0^a e^{-x}\,dx\right)\left(\int_0^\infty 2e^{-2y}\,dy\right)$$

$$= (1-e^{-a})\cdot 1 = 1-e^{-a}.$$

Ex. Disc of radius R. In it a point (x,y) is randomly chosen. In particular, the density function is given by

$$f(x,y) = \begin{cases} c & \text{if } x^2+y^2 \leq R^2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine c

(b) Find marginal density functions of X and Y.

(c) Compute $P(L \leq a)$ where L is the distance to the origin.

(d) Find $E[L]$.

Solution:

(a) $$c \iint_D f(x,y)\,dx\,dy = c \iint_D 1\,dx\,dy = c\pi R^2 = 1.$$

Hence $c = \frac{1}{\pi R^2}$.

(b) $$F_X(a) = P(X \le a) = P(X \le a, Y < \infty)$$

$$= \int_{-R}^{a} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\,dy\,dx =$$

$$= \int_{-R}^{a} 2\,\frac{1}{\pi R^2} \sqrt{R^2-x^2}\,dx$$

Thus $f_X(a) = \frac{\partial}{\partial a} F_X(a) = \frac{2}{\pi R^2}\sqrt{R^2-a^2}$

so $f_X(x) = \frac{2}{\pi R^2}\sqrt{R^2-x^2} \qquad -R \le x \le R$

Similarly $f_Y(y) = \frac{2}{\pi R^2}\sqrt{R^2-y^2} \qquad -R \le y \le R$.

(c) The simplest way to compute $P(L \leq a)$ is simply to observe that the probability $(x,y) \in A$ is proportional to the area. Thus if $A \subset D$

$$P((x,y) \in A) = \frac{\text{area}(A)}{\text{area}(D)}$$

Hence $P(L \leq a) = \dfrac{\pi a^2}{\pi R^2} = \left(\dfrac{a}{R}\right)^2$

Alternatively

$$P(L \leq a) = \iint_{(x,y):\, x^2+y^2 \leq a^2} \frac{1}{\pi R^2}\, dx\, dy = \int_0^{2\pi} \int_0^a \frac{1}{\pi R^2}\, r\, dr\, d\theta$$

$$= \frac{2\pi}{\pi R^2} \cdot \frac{1}{2} \pi a^2 = \left(\frac{a}{R}\right)^2$$

(d) $E[L] = \displaystyle\int_0^R a f_L(a)\, da$

where $f_L(a) = \dfrac{d}{da} F_L(a) = \dfrac{d}{da} P(L \leq a) = \dfrac{2a}{R^2}$

Hence $E[L] = \displaystyle\int_0^R \frac{2a^2}{R^2}\, da = \frac{2}{3} \cdot \frac{a^3}{R^2} \Big|_0^R = \frac{2}{3} R$

Alternatively,

$$E[L] = \iint_{(x,y):\, x^2+y^2 \leq R^2} \sqrt{x^2+y^2}\, \frac{1}{\pi R^2}\, dx\, dy = \int_0^{2\pi} \int_0^R \frac{r^2}{\pi R^2}\, dr\, d\theta$$

$$= \frac{2\pi}{\pi R^2} \cdot \frac{R^3}{3} = \frac{2}{3} R.$$

Ex. The joint density of X and Y is given by

$$f(x,y) = \begin{cases} e^{-(x+y)} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find density function of $Z = \frac{X}{Y}$

Solution:

$$F_Z(a) = P(Z \leq a) = P(X \leq aY) = \int_0^\infty \int_0^{ay} e^{-x} e^{-y} \, dx \, dy$$

$$= \int_0^\infty e^{-y} \int_0^{ay} e^{-x} \, dx \, dy = \int_0^\infty e^{-y} (1 - e^{-ay}) \, dy$$

$$= 1 + \left[\frac{e^{-(a+1)y}}{a+1} \right]_0^\infty = 1 - \frac{1}{a+1}$$

$$f_Z(a) = \frac{d}{da} F_Z(a) = \frac{1}{(a+1)^2} \ ; \qquad 0 < a < \infty.$$

Joint Probability distribution For n Random Variables

As expected the joint cumulative distribution function is defined by

$$F(a_1, a_2, a_3 \ldots, a_n) = P(X_1 \leq a_1, X_2 \leq a_2, X_3 \leq a_3 \ldots)$$

If the variables $X_1, \ldots, X_n$ are jointly continuous, there exists a joint probability density function $f(x_1, \ldots, x_n)$ such that for $C \subset \mathbb{R}^n$

$$p((x_1, x_2, \ldots, x_n) \in C) = \underset{(x_1, x_2, \ldots, x_n) \in C}{\iint \cdots \int} f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n$$

Ex. (Multinomial distribution). n independent trials are performed. Each trial has r outcomes $\{a_1, \ldots, a_r\}$ with $p(a_k) = P_k$; $\sum_{k=1}^{r} P_k = 1$

Define X_k = # of outcomes of type a_k. $1 \leq k \leq r$.

Then $P(X_1 = n_1, X_2 = n_2, \ldots, X_p = n_r)$

$$= \binom{n}{n_1\ n_2 \ldots n_r} P_1^{n_1} P_2^{n_2} \ldots P_r^{n_r} \ ; \quad \sum_{k=1}^{r} n_k = n.$$

<u>Ex.</u> Experiment consists in rolling a fair die 9 times. Each trial has 6 possible outcomes. Let $X_k =$ # of outcomes where the die landed on the value k, $k = 1, \ldots, 6$.

Then $P(X_1 = 0, X_2 = 0, X_3 = 4, X_4 = 1, X_5 = 2, X_6 = 2)$

$$= \binom{9}{0\,0\,4\,1\,2\,2} \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^4 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 = \frac{9!}{4!\,1!\,2!\,2!} \left(\frac{1}{6}\right)^9.$$