Continuous Random Variables: Lecture 6

Exponential Distribution

Def: A continuous random variable X is said to have the Exponential distribution with parameter λ, where $\lambda > 0$ if its pdf is

$$f(x) = \lambda e^{-\lambda x} \quad x > 0.$$

We denote this by $X \sim \text{Expo}(\lambda)$.

The corresponding cdf is

$$F(x) = 1 - e^{-\lambda x} \quad x > 0$$

It is sufficient to consider $X \sim \text{Expo}(1)$ to understand all other exponential distributions. To see this, suppose $Y = \frac{X}{\lambda}$. Then $P(Y \leq y) = P(\frac{X}{\lambda} \leq y) = P(X \leq \lambda y)$

$$= \int_0^{\lambda y} e^{-x} dx = -e^{-x} \Big|_0^{\lambda y} = 1 - e^{-\lambda y}$$

Hence $f_Y(y) = \frac{d}{dy}(1 - e^{-\lambda y}) = \lambda e^{-\lambda}$. In other words

$Y \sim \text{Expo}(\lambda)$.

On the other hand, if $Y \sim \mathcal{E}xpo(\lambda)$ then $X = \lambda Y$ satisfies $X \sim \mathcal{E}xpo(1)$ because. $P(X \leq x) =$

$$= P\left(Y \leq \frac{x}{\lambda}\right) = \int_0^{\frac{x}{\lambda}} \lambda e^{-\lambda y}\, dy = 1 - e^{-x}$$

Hence $f_X(x) = \frac{d}{dx}\left(1 - e^{-x}\right) = e^{-x}$.

<u>Expectation and variance of exponential r.v.</u>

Let $X \sim \mathcal{E}xpo(1)$ then

$$E[X] = \int_0^\infty x e^{-x}\, dx = -x e^{-x} \Big|_0^\infty + \int_0^\infty e^{-x}\, dx$$

$$= \int_0^\infty e^{-x}\, dx = -e^{-x} \Big|_0^\infty = 1.$$

$$E[X^2] = \int_0^\infty x^2 e^{-x}\, dx = -x^2 e^{-x} \Big|_0^\infty + 2\int_0^\infty x e^{-x}\, dx$$

$$= 2\int_0^\infty x e^{-x}\, dx = 2$$

Thus $Var(X) = E[X^2] - (E[X])^2 = 2 - 1 = 1$.

In general, if $Y \sim \mathcal{E}xpo(\lambda)$, $Y = \frac{X}{\lambda}$ so that

$$E[Y] = E\left[\frac{X}{\lambda}\right] = \frac{1}{\lambda} E[X] = \frac{1}{\lambda}$$

$$E[Y^2] = \left(\frac{1}{\lambda}\right)^2 E[X^2] = 2\left(\frac{1}{\lambda}\right)^2.$$

and therefore

$$Var(Y) = E[Y^2] - (E[Y])^2 = 2\left(\frac{1}{\lambda}\right)^2 - \left(\frac{1}{\lambda}\right)^2 = \left(\frac{1}{\lambda}\right)^2$$

The exponential distribution has a very special property called the memoryless property, which says that even if you've waited for hours or days without success, that success isn't any more likely to arrive soon. In fact, you might as well have just started waiting one moment ago:

<u>Ex.</u> Suppose the time it takes me to grade one of your tests is exponentially distributed with mean of 1 month. Given that you've waited for the results 1 month already, what is the probability that it will take another month or longer to grade the test?

<u>Solution:</u>

Let $T \sim Expo(1)$. Then we wish to compute

$$P(T \geq 2 \mid T \geq 1) = P(1+1 \mid T \geq 1) = \frac{P(T \geq 2, T \geq 1)}{P(T \geq 1)}$$

$$= \frac{P(T \geq 2)}{P(T \geq 1)} = \frac{1 - P(T<2)}{1 - P(T<1)} =$$

$$= \frac{1 - \int_0^2 e^{-x}\,dx}{1 - \int_0^1 e^{-x}\,dx} = \frac{1 + e^{-x}\Big|_0^2}{1 + e^{-x}\Big|_0^1}$$

$$= \frac{e^{-2}}{e^{-1}} = e^{-1} = P(T \geq 1).$$

The same probability you started when you handed in the test!

Def: (Memoryless property). A continuous distribution is said to have the memoryless property if a random variable from that distribution satisfies

$$P(X \geq s+t \mid X \geq s) = P(X \geq t).$$

For all $s, t \geq 0$.

That is, if you waited s minutes, the probability of waiting another t minutes or more is the same as if you haven't waited at all.

We can see that if $X \sim Expo(\lambda)$, then X satisfies the memoryless property.

$$P(X \geqslant s+t \,/\, X \geqslant s) = \frac{P(X \geqslant s+t)}{P(X \geqslant s)} =$$

$$= \frac{\int_{s+t}^{\infty} \lambda e^{-\lambda x}\, dx}{\int_{s}^{\infty} \lambda e^{-\lambda x}\, dx} = \frac{-e^{-\lambda x}\Big|_{s+t}^{\infty}}{-e^{-\lambda x}\Big|_{s}^{\infty}}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X \geqslant t).$$

Conversely if X has the memoryless property, $X \sim Expo(\lambda)$:

<u>Thm:</u> If X is a positive continuous r.v. with the memoryless property, then X has an Exponential distribution.

<u>Proof:</u> Let F be the cdf of X and define

$$G(x) = 1 - F(x) = P(X > x).$$

By the memoryless property $P(X > s+t \,|\, X > s) =$

$$= \frac{P(X > s+t)}{P(X > s)} = P(X > t) \quad \text{or}$$

$$\frac{G(s+t)}{G(s)} = G(t). \quad \text{for all } s,t \geqslant 0.$$

$G(s+t) = G(s)\,G(t)$.

Thus, in particular, for any integer $n \geq 0$

$$G(n) = G(n-1+1) = G(n-1)G(1) = \ldots \underset{1}{G(1)}\,\underset{2}{G(1)} \ldots \underset{n}{G(1)}$$

$$= G(1)^n$$

Similarly

$$G(1) = G\left(\underset{1}{\tfrac{1}{n}} + \underset{2}{\tfrac{1}{n}} + \ldots + \underset{n}{\tfrac{1}{n}}\right) = G\left(\tfrac{1}{n}\right)^n$$

Hence

$$G\left(\tfrac{1}{n}\right) = G(1)^{\frac{1}{n}}$$

In general, for any rational number $\frac{m}{n}$

$$G\left(\tfrac{m}{n}\right) = G\left(\underset{1}{\tfrac{1}{n}} + \underset{2}{\tfrac{1}{n}} + \ldots + \underset{m}{\tfrac{1}{n}}\right) = G\left(\tfrac{1}{n}\right)^m = \left[G(1)^{\frac{1}{n}}\right]^m$$

$$= G(1)^{\frac{m}{n}}.$$

Any continuous function is completely determined by its action on rational numbers. It follows that

$G(x) = G(1)^x$ for all $x > 0$. Thus $G(x) = e^{\ln(G(1))\,x}$

$= e^{-\ln\left(\frac{1}{G(1)}\right)x}$. Setting $-\lambda = -\ln\left(\frac{1}{G(1)}\right)$ it follows

that $\lambda > 0$ and $G(x) = e^{-\lambda x}$. Hence $F(x) = 1 - e^{-\lambda x}$

and $f(x) = \lambda e^{-\lambda x}$

Thus $X \sim \text{Expo}(\lambda)$ as desired.

It is hard to conceive that anything in nature is trully memoryless. As Aleksandr Gradinsky sings:

"Nothing on earth passes without a trace. And bygone youth is nonetheless immortal. How youthful we've been! How youthful..."

Nevertheless, exponential distributions find a variety of applications.

1. Some physical phenomena, such as radioactive decay, truly do exhibit the memoryless property.

2. The exponential distribution is well-connected to other named distributions. For instance, the exponential and Poisson distributions can be united by a shared story,

3. The exponential distribution serves as a building block for more flexible distributions such as the Weibull distribution, that allow for a wear-and-tear effect.

To understand these distributions, we first have to understand the Exponential.

Ex. I have a friend, Dave, who used to be very tardy. We often took very long walks that would start either at my place or his. When he scheduled to meet by his house at 12:00, I knew that the time it would take him to come out, D, is exponentially distributed with parameter $\frac{1}{10}$. I would therefore implement "Dave time correction"

specifically

(i) when he would schedule to meet by his house at 12:00, my arrivial time, A, would be uniformly distributed between 12:00 and 12:10.

(ii) when he would swing by my house at some time t, I would come out to greet him at a time uniformally distributed over $[t, t+10]$.

(a) Given that Dave has been waiting for me for the past 6 minutes (by my house), what is the probability he will have to wait another 3?

(b) Given that I was waiting for the past 10 minutes for Dave to come out of his house, what is the probability I will have to wait 10 or more additional minutes?

(c) Annoyed, I complain to Dave that he is very inconsiderate. He rebuffs this accusation with his typically cynical smug. "On average we wait for each other the same amount of time" – he says – "When I visit you, it takes you on average 5 minutes to come out. Whereas when I tell you to meet me by my house at 12:00, you typically arrive 5 minutes late and I exit my house, on average at 12:10. You wait for me on average 5 minutes. We're equal buddy!" Who is right?

Solution:

(a) $$P(A>3+6 \mid A>6) = \frac{P(A>9)}{P(A>6)} = \frac{\frac{1}{10}}{\frac{4}{10}}$$

$$= \frac{1}{4}.$$

(b) The exponential random variable D is memoryless so $$P(D>10) = \int_{10}^{\infty} \frac{1}{10} e^{-\frac{t}{10}} dt = -e^{-\frac{t}{10}} \Big|_{10}^{\infty} = e^{-1}$$

$\approx 0.37.$

(c) $E[A] = 5$ and $$E[D] = \int_0^{\infty} \frac{t}{10} e^{-\frac{t}{10}} dt =$$

$$= -te^{-\frac{t}{10}} \Big|_0^{\infty} + \int_0^{\infty} e^{-\frac{t}{10}} dt = 10$$

Thus, when he visits me, he waits on average for 5 minutes.

When I visit him, my average waiting time is

$$E[D-A] = E[D] - E[A] = 10 - 5 = 5.$$

On average he waits for me just as long as I wait for him.

However, the probability that he will need to wait for me longer than 10 minutes is $P(A>10)=0$. whereas the probability that I wait longer than 10 minutes is $P(D-A>10) = P(D>A+10) =$

$$= \int_0^{10} P(D>a+10)\,\frac{1}{10}\,da = \int_0^{10} \frac{1}{10} \int_{a+10}^{\infty} \frac{1}{10} e^{-\frac{t}{10}}\,dt\,da$$

$$= \int_0^{10} \frac{1}{10} e^{-\frac{(a+10)}{10}}\,da = -e^{-\frac{(a+10)}{10}}\Big|_0^{10} = e^{-1} - e^{-2}$$

$$= 0.23.$$

Exponential random variable he is! Never remembers my birthday, the bastard!