Random Variables Lecture 4

Properties of expected Value

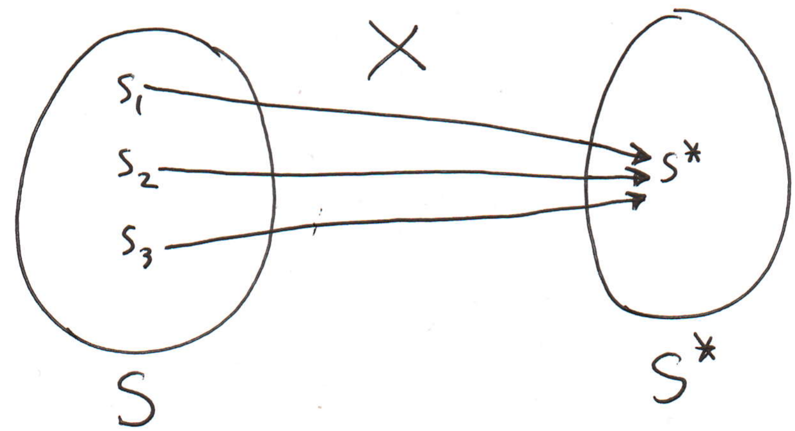
Recall:

1) X is a random variable if $X : S \mapsto S^*$ is a function between two samples spaces S and S^* s.t.

(a) $X(S) = S^*$

(b) $P(X = s^*) = \sum_{s \in S : X(s) = s^*} p(s)$ for every $s^* \in S^*$.

e.g.



$$P(X = s^*) = p(s_1) + p(s_2) + p(s_3)$$

2) If $S^* \subset \mathbb{R}$ $\quad E[X] = \sum_{s \in S} X(s)\,p(s)$ (why?)

3) $E[g(X)] = \sum_{s \in S} g(X(s))\,p(s)$

Thm: If X and Y are random variables over the same sample space S, $E[X+Y] = E[X] + E[Y]$

Proof: Let $Z = X + Y$

$$E[Z] = \sum_{s \in S} Z(s)p(s) = \sum_{s \in S} \big(X(s) + Y(s)\big)p(s)$$

$$= \sum_{s \in S} \big(X(s)p(s) + Y(s)p(s)\big) = \sum_{s \in S} X(s)p(s) + \sum_{s \in S} Y(s)p(s)$$

$$= E[X] + E[Y].$$

Corollary: $E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$

Ex. (a) Two fair dice are rolled. Compute the expected value of the sum.

(b) What is the expected value when n dice are rolled?

Solution:

(a) Let X be the number rolled on the first die and Y - the number on the second die.

Then $Z = X + Y$ is the sum.

$$E[Z] = E[X+Y] = E[X] + E[Y] = 2E[X] =$$

$$= 2\sum_{k=1}^{6} k\frac{1}{6} = 2 \cdot \frac{6 \cdot 7}{2} \cdot \frac{1}{6} = 7.$$

(b) Let X_k be the value of the k^{th} die, $k=1,\dots,n$

$$E[X_1+\dots+X_n] = E[X_1]+\dots+E[X_n] = nE[X_1] = n\cdot\frac{7}{2}.$$

Ex. An experiment consists of n independent trials. Each trial is a success with probability p. Find the expected number of successes.

Solution: (a) We may think intuitively as follows: If we perform great many trials m, by our limit understanding of probability

$$p = \lim_{m\to\infty} \frac{m(\text{success})}{m}$$

where $m(\text{success})$ = # of trials out of m that resulted in the successful outcome. When m is large, $p \approx \frac{m(\text{success})}{m}$ or $mp \approx m(\text{success})$.

Now suppose the experiment is repeated a large number of times k. Then the total number of trials $= k\cdot n$.

Among these trials we expect $\approx knp$ successes.

In particular, the expected number of successes per experiment

$$\approx \frac{\text{total successes}}{\text{number of experiments}} = \frac{knp}{k}$$

Thus the desired number is $\lim_{k\to\infty} \frac{knp}{k} = np.$

(b) Let $X_k = \begin{cases} 1 & \text{if trial } k \text{ is success} \\ 0 & \text{otherwise.} \end{cases}$

Then the expected number of successes is

$$E[X_1 + \ldots + X_n] = E[X_1] + \ldots + E[X_n] = n(1 \cdot p + 0 \cdot (1-p))$$

$= np.$

<u>Ex.</u> A new drug is tested. Out of 10 men and 10 women volunteers a sample of 10 people is randomly chosen to participate in the trials. What is the expected number of men in the chosen sample?

<u>Solution:</u> (a) Intuitively we expect half of the participants to be men. Hence $\frac{1}{2} \cdot 10 = 5$.

(b) Let $X = 0, 1, \ldots, 10$ be a random variable indicating the number of men.

Then

$$E[X] = \sum_{k=0}^{10} \frac{\boxed{k\binom{10}{k}}\binom{10}{10-k}}{\binom{20}{10}}$$

$$= \sum_{k=1}^{10} \frac{\boxed{10\binom{9}{k-1}}\binom{10}{10-k}}{\binom{20}{10}} \stackrel{j=10-k}{=} \sum_{j=0}^{9} \frac{10\binom{9}{9-j}\binom{10}{j}}{\binom{20}{10}}$$

$$\boxed{\sum_{j=0}^{n} \binom{n}{n-j}\binom{m}{j} = \binom{n+m}{n}}$$

$$= \frac{10\binom{19}{9}}{\binom{20}{10}} = \frac{1}{2} \frac{20\binom{19}{9}}{\binom{20}{10}}$$

$$= \frac{1}{2} \frac{10\binom{20}{10}}{\binom{20}{10}} = 5.$$

(c) Let $X_k = \begin{cases} 1 & \text{if } k^{th} \text{ person chosen is a man} \\ 0 & \text{otherwise} \end{cases}$

$$E[X] = E\left[\sum_{k=1}^{10} X_k\right] = \sum_{k=1}^{10} E[X_k] = 10\, E[X_1]$$

$$= 10 \cdot \frac{10}{20} = 5.$$

Make sure you understand this!

Ex. (Matching problem) n people write their names on a card, throw it in a bowl and pick a card at random. On the average how many people will select their own name?

Solution: Let $X_k = \begin{cases} 1 & \text{if } k^{th} \text{ person picks own name} \\ 0 & \text{otherwise.} \end{cases}$

$E[X_k] = \frac{1}{n}$ (why?) Hence $E[X_1 + \dots + X_n] =$

$= n\,E[X_1] = 1.$

Ex. Urn contains n balls k of which are blue and $n-k$ red. If a sample $m \leq n$ is randomly drawn, what is the expected number of blue balls?

Solution: Let $X_j = \begin{cases} 1 & \text{if } j^{th} \text{ ball is blue} \\ 0 & \text{otherwise.} \end{cases}$

We want $E[X_1 + \dots + X_m] = \sum_{j=1}^{m} E[X_j]$

Notice that $E[X_j] = p(j^{th} \text{ ball is blue}) = \frac{k}{n}$

Thus the expected number of blue balls in sample is $\frac{mk}{n}$

Ex. Urn initially contains 1 red and 1 blue ball. A ball is randomly extracted and then replaced by 2 balls of the same color. Let X - first time blue ball removed.

$X = 1, 2, 3, \ldots$

(a) Compute $P(X = k)$

(b) Compute $E[X]$

Solution:

(a) $P(X=k) = P(\text{red on first } k-1 \text{ trials \& blue on } k^{th} \text{ trial})$

$$= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{k-1}{k} \cdot \frac{1}{k+1} = \frac{(k-1)!}{(k+1)!} = \frac{1}{k(k+1)}$$

Probability blue ball is eventually drawn =

$$= \sum_{k=1}^{\infty} P(X=k) = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \lim_{N \to \infty} \left(\sum_{k=1}^{N} \frac{1}{k} - \sum_{k=1}^{N} \frac{1}{k+1} \right) = \lim_{N \to \infty} \left(1 - \frac{1}{N+1} \right) = 1.$$

(b) However

$$E[X] = \sum_{k=1}^{\infty} k \cdot \frac{1}{(k+1)k} = \sum_{k=1}^{\infty} \frac{1}{k+1} = \infty \quad !!!$$

what!? So we will draw a blue ball with probability 1 after finitely many steps, but it will take us ∞ many steps on average to do that.

Ex. A fair coin is tossed 16 times: What is the average number of times the pattern HTTH will be observed?

This problem might be a bit challenging. Take your time before moving on to the next page for the solution.

Solution: For $k = 1, 2, \ldots, 16-4+1 = 13$ let

$$X_k = \begin{cases} 1 & \text{if } k^{th} \text{ through } k+3^{th} \text{ tosses are HTTH} \\ 0 & \text{otherwise} \end{cases}$$

Then $E[X_k] = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$ (why?)

The desired number is $E\left[\sum_{k=1}^{13} X_k\right] = \sum_{k=1}^{13} E[X_k]$

$$= \sum_{k=1}^{13} \frac{1}{16} = \frac{13}{16}$$

In general, if coin is tossed n times, the expected number of words HTTH would be $\frac{n-3}{16}$.