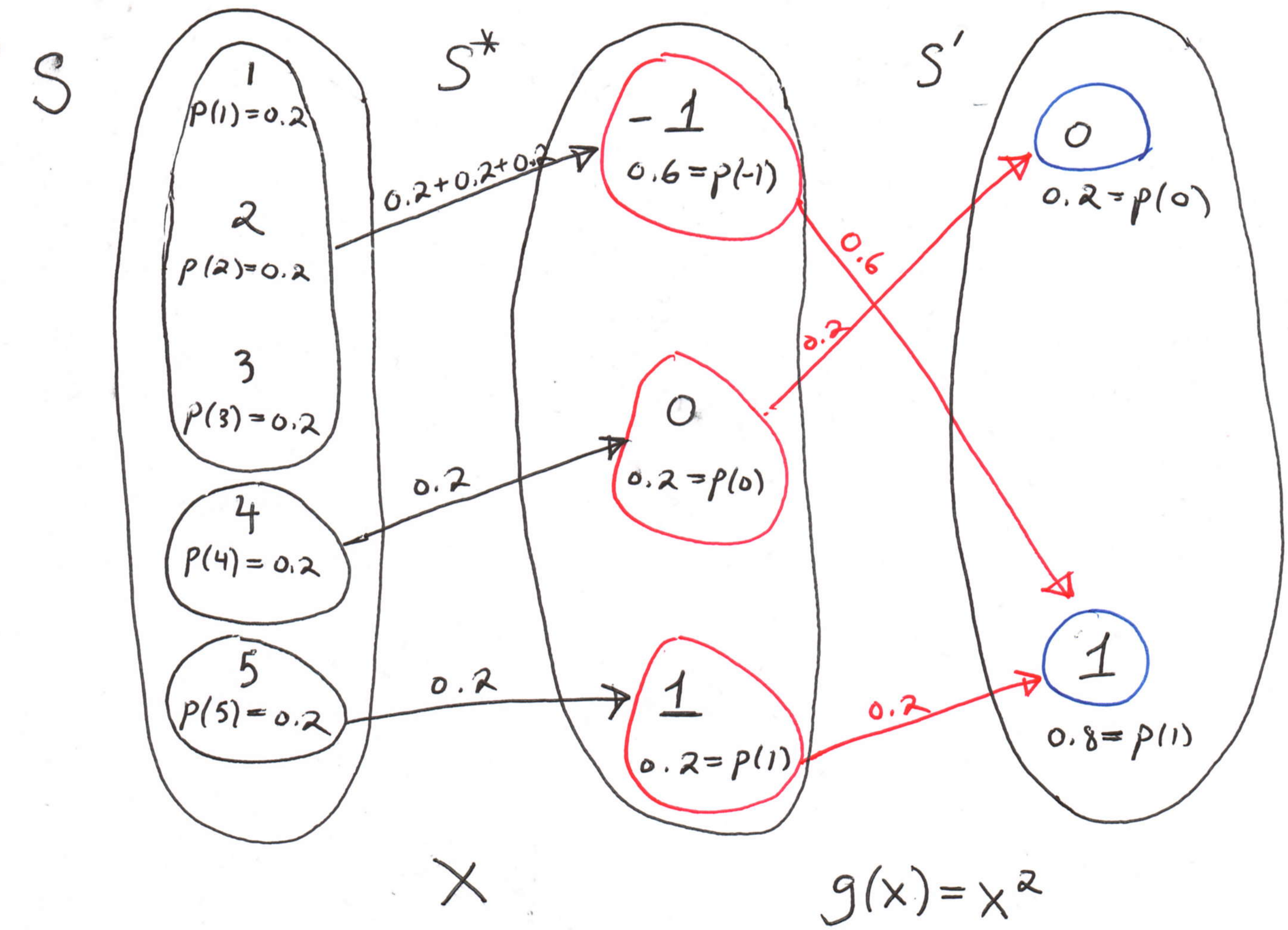
Random Variables Lecture 3

Expectation of a function of a random variable

Recall that a random variable is a function that collapses some sample space S to a sample space S^*, by associating outcomes in S having a common feature. There are times when we wish to further collapse S^* to S'



Observe that $P(X=-1) = 3\cdot 0.2 = 0.6$, $P(X=0) = 0.2$, and $P(X=1) = 0.2$

Now $P(X^2=0) = 0.2$ and $P(X^2=1) = 0.6+0.2 = 0.8$.

Note that $E[X^2] = 0\cdot P(X^2=0) + 1\, P(X^2=1) =$
$= 0.8$ while $(E[X])^2 = (-1\cdot 0.6 + 0\cdot 0.2 + 1\cdot 0.2)^2 =$
$= 0.16$

Hence $\boxed{E[X^2] \neq (E[X])^2}$

Notice that we computed $E[X^2]$ by first establishing the probabilities of the distinct values of X^2.

The sum $(-1)^2\cdot 0.6 + 0^2\cdot 0.2 + 1^2\cdot 0.2$ also equals to $E[X^2]$. To emphasize, by definition, if $Y = X^2$, $Y = 0$ or 1

$$E[Y] = 0\cdot p(Y=0) + 1\cdot p(Y=1) = 0\cdot 0.2 + 1\cdot 0.8 = 0.8$$
$$= 0\cdot 0.2 + 1^2\cdot 0.2 + (-1)^2\cdot 0.6.$$

This situation is true in general, as the following thm justifies.

Proposition: If X is a discrete random variable that takes on one of the values x_k, $k \geq 1$ with probability $p(x_k)$ then for any real-valued function g for which $\sum_{k=1}^{\infty} |g(x_k)| p(x_k) < \infty$

$$E[g(X)] = \sum_{k=1}^{\infty} g(x_k) p(x_k)$$

Proof: This has to do with rearranging the sum of an infinite series. The sum of a series that does not converge absolutely can be changed if we change the order in which the terms of the series are added up.

For example

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

can be rearranged from a finite sum to infinity.

However, all rearrangements of an absolutely convergent series yield the same sum.

Let $Y = g(X)$ and y_j, $j \geq 1$ be its distinct values.

Then $E[g(X)] = E[Y] = \sum_{j=1}^{\infty} y_j \sum_{k : g(x_k) = y_j} p(x_k)$

$$= \sum_{j=1}^{\infty} \sum_{k:g(x_k)=y_j} g(x_k)p(x_k) = \sum_{k=1}^{\infty} g(x_k)p(x_k)$$

because $\sum_{j=1}^{\infty} \sum_{k:g(x_k)=y_j} g(x_k)p(x_k)$ is a rearrangement of the latter series.

Ex. To illustrate the proposition, consider

$X = \pm 1, \pm 2, \ldots, \pm n; \quad p(X=n) = p(X=-n) = \left(\frac{1}{2}\right)^{|n|+1}.$

and let $Y = |X|$.

By def. of expected value

$$E[Y] = \sum_{n=1}^{\infty} n \sum_{k:|k|=n} p(k) = \sum_{n=1}^{\infty} n\left(\underbrace{\left(\frac{1}{2}\right)^{|n|+1}}_{k=-n} + \underbrace{\left(\frac{1}{2}\right)^{n+1}}_{k=n}\right)$$

$$= \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n} = 2$$

And by the proposition

$$E[Y] = \sum_{n=-\infty}^{-1} |n|\left(\frac{1}{2}\right)^{|n|+1} + \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n+1}$$

$$= 2\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n+1} = 2.$$

<u>Corollary:</u> If a and b are constants, then $E[aX+b] = a\,E[X]+b$

<u>Proof:</u> $E[aX+b] = \sum_{x:p(x)>0} (ax+b)p(x) = a\sum_{x:p(x)>0} x\,p(x) + b\sum_{x:p(x)>0} p(x) = aE[X] + b\cdot 1$.

<u>Variance</u>

Knowing the average payoff per game, $E[X]$, gives little information about how scores jump from game to game. For example

$$W = 0 \quad \text{with prob. } 1$$

$$Y = \begin{cases} -1 & \text{with prob. } \frac{1}{2} \\ 1 & \text{with prob. } \frac{1}{2} \end{cases}$$

$$Z = \begin{cases} -100 & \text{with prob. } \frac{1}{2} \\ 100 & \text{with prob. } \frac{1}{2}. \end{cases}$$

all have the same expected value 0. However W does not jump around from game to game, whereas Z undergoes the most extreme oscilations from among the three veriables.

Given a random variable X, it is helpful to know, on average, how far away are the values of X away from the expected value of X? Letting $\mu = E[X]$ we may try to compute $E[|X-\mu|]$. However, this calculation is often difficult so that $E[(X-\mu)^2]$ is used instead.

<u>Def:</u> Let X be a discrete random variable with $\mu = E[X]$
Then the variance of X, $Var(X) = E[(X-\mu)^2]$.

Observe that $Var(X) = E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2]$

$$= \sum_{x:p(x)>0} (x^2 - 2\mu x + \mu^2) p(x) = \sum_{x:p(x)>0} x^2 p(x) - 2\mu \sum_{x:p(x)>0} x\, p(x)$$

$$+ \mu^2 \sum_{x:p(x)>0} p(x) = E[X^2] - 2\mu E[X] + \mu^2$$

$$= E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - (E[X])^2$$

<u>Ex.</u> Let X be the outcome of tossing a die. Calculate $Var(X)$.

<u>Solution:</u> $E[X] = \sum_{k=1}^{6} \frac{k}{6} = \frac{7}{2}$. $E[X^2] =$

$$= \sum_{k=1}^{6} \frac{k^2}{6} = \frac{6 \cdot 7 \cdot 13}{6} \cdot \frac{1}{6} = \frac{91}{6}$$

$$Var(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}.$$

Useful Observation: $Var(aX+b) = a^2\, Var(X)$

Proof: Let $\mu = E[X]$. Then $E[aX+b] = a\,E[X] + b$

$= a\mu + b$

$$Var(aX+b) = E[(aX+b-a\mu-b)^2] = E[(aX-a\mu)^2] =$$

$$= a^2 E[(X-\mu)^2] = a^2\, Var(X).$$