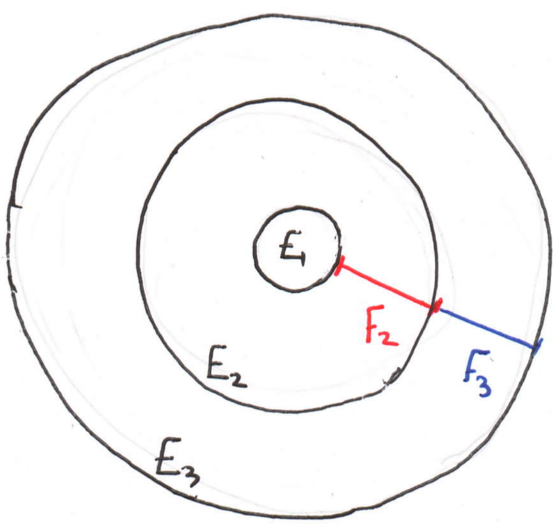
Random Variables Lecture 2

Properties of the cumulative distribution function

Lemma 1: Let $P : S \mapsto [0,1]$ be a probability function and $E_1 \subset E_2 \subset \ldots \subset E_n \subset \ldots$ a sequence of events in S, with $E = \bigcup_{n=1}^{\infty} E_n$. Then $P(E) = \lim_{n \mapsto \infty} P(E_n)$.

Proof: Define $F_1 = E_1$, $F_2 = E_2 - E_1$, $F_3 = E_3 - E_3$, etc.



The E's are discs and the Fs are annuli.

Observe that $\bigcup_{n=1}^{N} E_n = \bigcup_{n=1}^{N} F_n$ and the sequence $F_1, F_2, \ldots, F_N$ consists of disjoint sets. (for every integer N)

Thus, $P(E) = P\left(\bigcup_{n=1}^{\infty} E_n\right) = P\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} P(F_n) =$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} P(F_n) = \lim_{N \to \infty} P\left(\bigcup_{n=1}^{N} F_n\right) = \lim_{N \to \infty} P\left(\bigcup_{n=1}^{N} E_n\right)$$

$$= \lim_{N\to\infty} P(E_N) = \lim_{n\to\infty} P(E_n)$$

<u>Lemma 2:</u> Let $P: S \longmapsto [0,1]$ be a probability function and $E_1 \supset E_2 \supset \ldots$ be a sequence of decreasing events in S with $E = \bigcap_{n=1}^{\infty} E_n$. Then $P(E) = \lim_{n\to\infty} P(E_n)$.

<u>Proof:</u> Observe that $E^c = \bigcup_{n=1}^{\infty} E_n^c$ and $E_1^c \subset E_2^c \subset \ldots$

Hence by lemma 1 $P(E^c) = \lim_{n\to\infty} P(E_n^c) = \lim_{n\to\infty} 1 - P(E_n)$.

so $$1 - P(E) = 1 - \lim_{n\to\infty} P(E_n)$$

$$P(E) = \lim_{n\to\infty} P(E_n)$$

If X is a random variable, recall that the cumulative distribution function F_X is defined by $F_X(b) = P(X \leq b)$

F_X has the following properties:

<u>1. If $a < b$, $F_X(a) \leq F_X(b)$.</u>

This follows because $\{X \leq a\} \subset \{X \leq b\}$

<u>2. $\lim_{b\to\infty} F_X(b) = 1$</u>

Setting $E_N = \{X \leq N\}$, $E_1 \subset E_2 \subset \ldots \subset \mathbb{R}$ with $\mathbb{R} = \bigcup_{N=1}^{\infty} E_N$

By lemma 1, $1 = P(\mathbb{R}) = \lim_{N\to\infty} P(E_N) = \lim_{N\to\infty} F_X(N)$

3. $\lim_{b \to -\infty} F(b) = 0$

Setting $E_N = \{X \leq -N\}$, $E_1 \supset E_2 \supset \cdots \supset \emptyset$. and $\bigcap_{N=1}^{\infty} E_N = \emptyset$. It follows from lemma 2 that

$$0 = P(\emptyset) = \lim_{N \to \infty} P(E_N) = \lim_{N \to \infty} F_X(-N).$$

All probability questions about X can be answered in terms of c.d.f. For example

$$P(a < X \leq b) = P(\{X \leq b\} - \{X \leq a\}) = P(\{X \leq b\} \cap \{X \leq a\}^c)$$
$$= P(X \leq b) - P(\{X \leq b\} \cap \{X \leq a\}) = P(X \leq b) - P(X \leq a)$$
$$= F_X(b) - F_X(a).$$

If we want to compute $P(X < b)$, we simply observe

$$P(X < b) = P\left(\lim_{n \to \infty} \{X \leq b - \tfrac{1}{n}\}\right) = \lim_{n \to \infty} P(X \leq b - \tfrac{1}{n})$$
$$= \lim_{n \to \infty} F(b - \tfrac{1}{n})$$

Ex. The c.d.f. of X is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \leq x < 1 \\ \frac{2}{3} & 1 \leq x < 2 \\ \frac{11}{12} & 2 \leq x < 3 \\ 1 & 3 \leq x \end{cases}$$

Compute

(a) $P(X<2)$

(b) $P(X=1)$

(c) $P(X>\frac{1}{2})$

(d) $P(2<X\leqslant 4)$

<u>Solution:</u>

(a) $P(X<3) = \lim\limits_{n\to\infty} P(X \leqslant 3-\frac{1}{n}) = \lim\limits_{n\to\infty} F_X(3-\frac{1}{n})$

$= \frac{11}{12}$

(b) $P(X=1) = P(X \leqslant 1) - P(X<1) = F_X(1) - \lim\limits_{n\to\infty} F_X(1-\frac{1}{n})$

$= \frac{2}{3} - \lim\limits_{n\to\infty} \frac{1-(\frac{1}{n})}{2} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

(c) $P(X>\frac{1}{2}) = 1-P(X\leqslant\frac{1}{2}) = 1-F_X(\frac{1}{2}) = 1-\frac{(\frac{1}{2})}{2}$

$= 1-\frac{1}{4} = \frac{3}{4}$

(d) $P(2<X\leqslant 4) = P(\{X\leqslant 4\}\cap\{X\leqslant 2\}^c)$

$= P(X\leqslant 4) - P(\{X\leqslant 4\}\cap\{X\leqslant 2\}) =$

$= P(X\leqslant 4) - P(X\leqslant 2) = F_X(4) - F_X(2) = 1-\frac{11}{12}$

$= \frac{1}{12}.$

St. Petersburg paradox

Fair coin is tossed until it lands on heads. If the coin lands on heads on the n^{th} trial you are paid 2^n dollars. Would you wage \$1000,000 to play this game? How about this: You are given unlimited credit. You must pay your debt only when you decide to quit the game.

Solution: Recall that expected value represents the average payoff per game as the number of games played tends to infinity.

Let $X = 2^1-10^6, 2^2-10^6, \ldots, 2^n-10^6, \ldots$ be the payoffs for this game. Then $p(X=2^n-10^6) = \frac{1}{2^n}$

Thus $E[X] = \sum_{n=1}^{\infty}(2^n-10^6)\frac{1}{2^n} = \sum_{n=1}^{\infty}\left(1-\frac{10^6}{2^n}\right)$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} 1 - \sum_{n=1}^{N}\frac{10^6}{2^n} = \lim_{N\to\infty} N - 10^6\left(1-\left(\tfrac{1}{2}\right)^N\right)$$

$= \infty$. The average payoff is $\$\infty$!!!

So? How about it?

Modified St Petersburg

Some of you are worried about the non-existence of $\$\infty$ (They are printing money. Don't you worry!!) To make the game a little more real, suppose that the coin will be tossed until either heads comes up or a total of N tosses is made. If heads comes up on the n^{th} toss ($n \leq N$) you are payed $\$2^n$. If no heads appears, you are payed $\$2^{N+1}$. How large must N be to make the expected payoff positive?

Solution: $X = 2^n - 10^6$; $P(X = 2^n - 10^6) = \left(\frac{1}{2}\right)^n$ if $n \leq N$ and $p(X = 2^{N+1} - 10^6) = 1 - \sum_{n=1}^{N} \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^N$.

$$E[X] = \sum_{n=1}^{N} (2^n - 10^6)\left(\frac{1}{2}\right)^n + (2^{N+1} - 10^6)\left(\frac{1}{2}\right)^N$$

$$= N + 2 - 10^6 \geq 1$$

$$\Longrightarrow N \geq 10^6 - 1.$$

Thus, to make the game the least bit profitable, the coin must be allowed to land on heads on 10^6-1^{th} toss for the first time!

Thus we must allow for probabilities as low as $\left(\frac{1}{2}\right)^{10^6-1}$. How unlikely is the event that a coin lands on heads on the 10^6-1^{th} trial for the first time? Well, the number of atoms in the entire universe is estimated at 10^{82}. $2^{10^6-1} > 10^{200,000}$!!!

Even if you were to shrink the universe to the size of an atom there would be far too many atoms for ~~2439~~ $10^{200,000-82}$ universes.