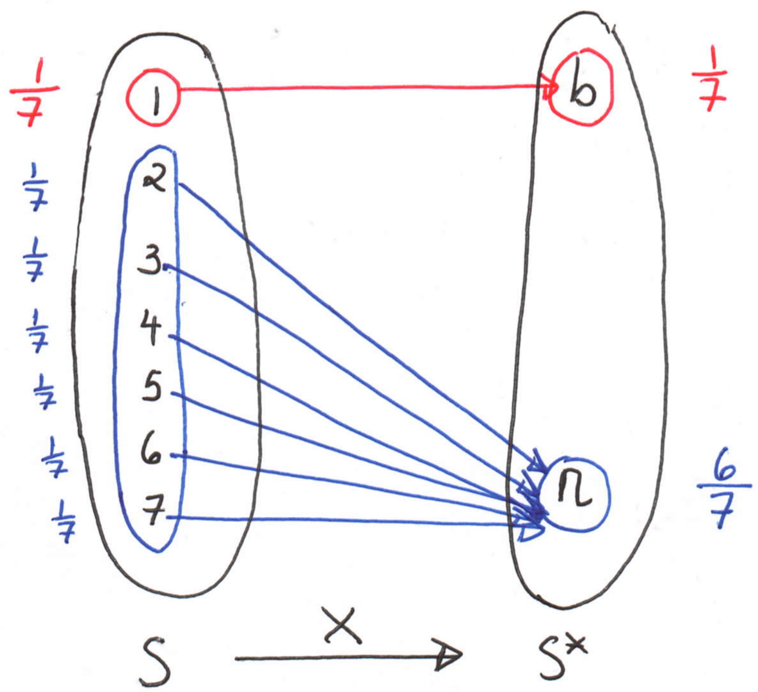
Random Variables Lecture 1

- Generate an induced sample space.
  - Transition from given sample space to one of more significant interest.

For example, when we play Russian roulette with 7-chamber pistol, given sample space $S = \{1,2,3,4,5,6,7\}$ where outcome is chamber in line of fire.
Induced Sample space is $S^* = \{b, n\}$ (bullet, no bullet).



- X maps 1 to b and 2,…,7 to n.

- Think of outcomes in S as pebbles and of the probabilities of these outcomes as the weights of these stones. X crushes these stones together, adding corresponding weights.

Def. Let $X : S \xrightarrow{\text{onto}} S^*$ be a surjective function from a sample space S onto a set S^*. Then X is a random variable if X induces a probability on S^* by $P(y) = P(X^{-1}(y)) = \sum_{x \in S : X(x) = y} p(x)$

In the previous example, this just means

$$p(b) = p(X^{-1}(b)) = p(1) = \frac{1}{7}$$

$$p(n) = p(X^{-1}(n)) = \sum_{k=2}^{7} p(k) = \frac{6}{7}.$$

Ex. 3 balls randomly selected without replacement from urn containing 20 balls numbered 1-20. If we bet that at least one of the chosen balls has a number as large as or larger that 17, what's the probability that we win the bet?

Solution: Let $X = 3, 4, \ldots, 20$ be the largest denomination of the 3 chosen balls. We want

$$P(X \geq 17) = p(X=17) + p(X=18) + p(X=19) + p(X=20)$$

$$= \frac{\binom{16}{2} + \binom{17}{2} + \binom{18}{2} + \binom{19}{2}}{\binom{20}{3}} \approx 0.508$$

<u>Ex.</u> A fair die is tossed until it comes up 6. Let X denote the number of required tosses.

Then $X = 1, 2, 3, \dots, n, \dots, \infty$.

$$P(X=1) = \frac{1}{6}$$

$$P(X=2) = \frac{5}{6} \cdot \frac{1}{6}$$

$$\vdots$$

$$P(X=n) = \left(\frac{5}{6}\right)^{n-1} \frac{1}{6}$$

The probability that finitely many tosses is required

$$P(X \in \mathbb{N}) = P\left(\bigcup_{n=1}^{\infty} \{X=n\}\right) = \sum_{n=1}^{\infty} P(X=n) =$$

$$= \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^{n-1} \frac{1}{6} = \frac{1}{1-\frac{5}{6}} \cdot \frac{1}{6} = \frac{1}{6-5} = 1.$$

Thus $P(X=\infty) = 1 - P(X \in \mathbb{N}) = 1 - 1 = 0$.

<u>Q.</u> Does it mean ∞ cannot happen?

<u>Discrete Random Variables</u>

Random variables that take on at most a countable number of possible values are said to be discrete.

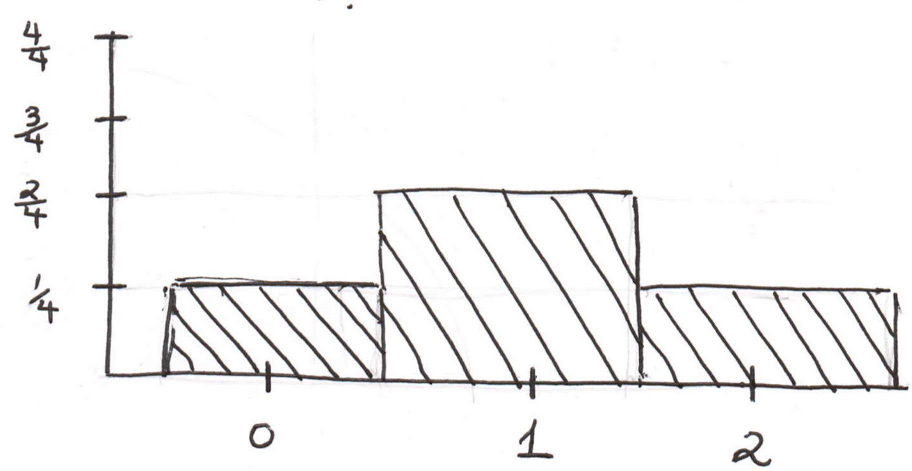
If $x_1, x_2, \ldots,$ are the values that X can take with nonzero probability, then $p(x_n) \geq 0$ for $n=1,2,\ldots,$ is called $\boxed{\text{probability mass function pmf}}$

Since X must take on one of the values x_n, we have
$$\sum_{n=1}^{\infty} p(x_n) = 1.$$

Ex. Fair coin is tossed 2 times. Let X be the number of heads obtained. Then $X = 0, 1, 2$.

$p(0) = p(\{T,T\}) = \frac{1}{4}$, $p(1) = p(\{TH, HT\}) = \frac{1}{2}$.

$p(2) = p(\{HH\}) = \frac{1}{4}$.



Ex. The probability mass function of a random variable X is given by $p(n) = c\,\frac{\lambda^n}{n!}$ $n=0,1,2,\ldots$ where λ is some positive value. Find

(a) $P(X=0)$ and

(b) $P(X>2)$

Solution: Since X must take one of the values $0,1,2,\ldots,$

$$1 = \sum_{n=0}^{\infty} c\,\frac{\lambda^n}{n!} = c\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} = ce^{\lambda}.$$ Thus $c=e^{-\lambda}$

(a) $P(X=0) = e^{-\lambda}\,\frac{\lambda^0}{0!} = e^{-\lambda}.$

(b) $$P(X>2) = 1-P(X\leq 2) = 1-\Big(P(X=0)+P(X=1)+P(X=2)\Big) = 1-\left(e^{-\lambda}+e^{-\lambda}\frac{\lambda}{1}+e^{-\lambda}\frac{\lambda^2}{2}\right)$$
$$=1-e^{-\lambda}\left(1+\lambda+\frac{\lambda^2}{2}\right)$$

Cumulative Distribution

Let X be a discrete random variable and $a \in \mathbb{R}$.
We are often interested in problems of the form
$P(X \leq a) = F_X(a)$.

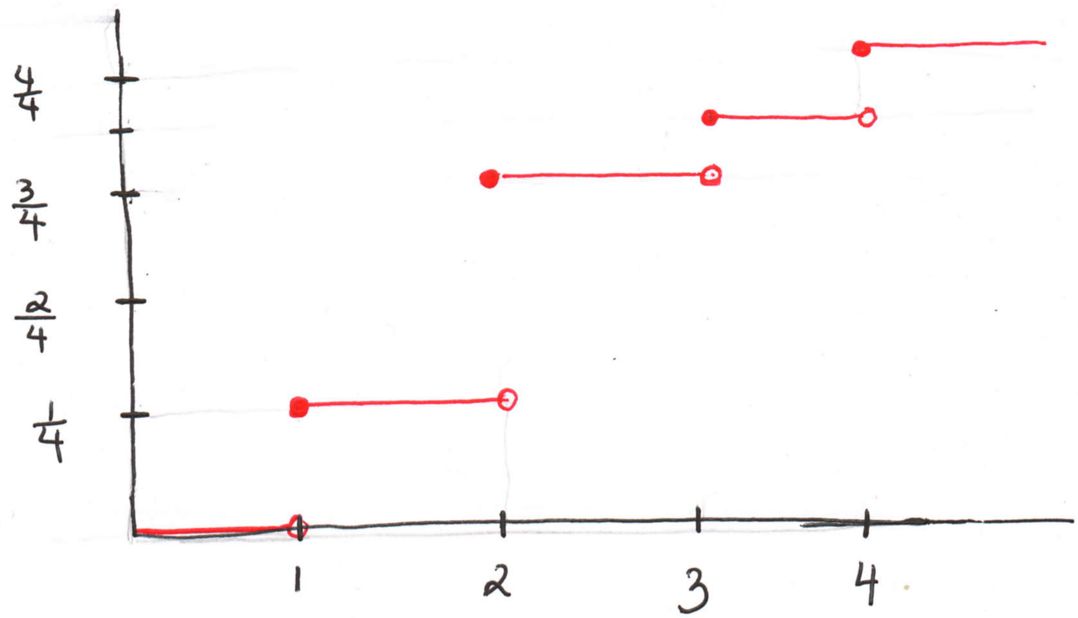
If X takes on the values $x_1, x_2, \ldots$ then

$$F_X(a) = \sum_{n : x_n \leq a} p(x_n)$$

Ex. Suppose X has probability mass function
$p(1) = \frac{1}{4}$, $p(2) = \frac{1}{2}$, $p(3) = \frac{1}{8}$, $p(4) = \frac{1}{8}$.

Then

$$F_X(a) = \begin{cases} 0 & \text{if} \quad a < 1 \\ \frac{1}{4} & \text{if} \quad 1 \leq a < 2 \\ \frac{1}{4} + \frac{1}{2} & \text{if} \quad 2 \leq a < 3 \\ \frac{1}{4} + \frac{1}{2} + \frac{1}{8} & \text{if} \quad 3 \leq a < 4 \\ \frac{1}{4} + \frac{1}{2} + \frac{1}{8} + \frac{1}{8} & \text{if} \quad 4 \leq a \end{cases}$$



Cumulative distributions will become important when we speak about continuous random variables.

Expected Value

Q. what happens when we play a game of chance habitually?

- If a game of chance has payoffs $x_1, \ldots, x_k$ with probabilities $p(x_1), \ldots, p(x_k)$, what would be the average payoff per game if a large number of games, n, is played?

A. $X = x_1, \ldots, x_k$

$$\frac{x_1\, n(X=x_1) + x_2\, n(X=x_2) + \ldots + x_k\, n(X=x_k)}{n}$$

$$= \sum_{j=1}^{k} \frac{x_j\, n(X=x_j)}{n}$$ where $n(X=x_j)$ is the number of games among n that resulted in payoff x_j.

Then $$\lim_{n \to \infty} \sum_{j=1}^{k} x_j \frac{n(X=x_j)}{n} = \sum_{j=1}^{k} x_j \lim_{n \to \infty} \frac{n(X=x_j)}{n}$$

which equals ...

$$= \sum_{j=1}^{K} x_j \, p(x_j)$$

Def: Let X be a discrete random variable with values $x_1, x_2, \ldots$ Then the expected value of X,

$$E[X] = \sum_{j=1}^{\infty} x_j \, p(x_j)$$

$$= \sum_{x:\, p(x)>0} x \, p(x)$$

This represents the weighted average of the values of X.

Ex. Find $E[X]$, where X is the outcome when we roll a fair die?

Solution: $E[X] = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6}$

$$= \frac{1}{6}\sum_{k=1}^{6} k = \frac{1}{6}\cdot\frac{6\cdot 7}{2} = \frac{7}{2}.$$

Ex. Suppose that when a fair die is rolled, you win \$6 when die comes up 6 and lose \$1 if the die comes up with another number. Would you like to play this game? How about 1000,000 games?

Solution: Let $Y = -1, 6$ be the possible payoffs.

$$E[Y] = -1 \cdot \frac{5}{6} + 6 \cdot \frac{1}{6} = 1 - \frac{5}{6} = \frac{1}{6}.$$

Thus, if you play many games, the outcome will be the same as if you get $\$\frac{1}{6}$ per game.

If you play 1000,000 games, your total payoff will be $\$\frac{1000{,}000}{6}$.

Ex. A game of Russian roulette is played with a revolver with 7 chambers. How many rounds do we expect the game to last?

Solution: We can solve this problem with just a little intuition.

The bullet is in one chamber, so the pistol fires once every...

yes exactly! Every 7 rounds! Thus we expect the gun to fire once every 7 rounds on the average, making the tournament's duration typically 7 rounds long.

Lets solve this rigorously.

$X = 1, 2, 3 \ldots$ be the number of rounds.

$$p(X=n) = \underbrace{\left(\frac{6}{7}\right)^{n-1}}_{n-1 \text{ misfires}} \underbrace{\frac{1}{7}}_{\text{fires on the last round.}}$$

$E[X] = \sum_{n=1}^{\infty} n \left(\frac{6}{7}\right)^{n-1} \frac{1}{7}$. This series reminds me of the movie Deer Hunter.

<u>Recall:</u> $G = \sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \ldots +$

$= 1 + x \underbrace{(1 + x + \ldots +)}_{G} = 1 + xG \; ; \; G = \frac{1}{1-x}$.

$$H = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} \left[(n-1) x^{n-1} + x^{n-1}\right] = \sum_{n=1}^{\infty} (n-1) x^{n-1} + \frac{1}{1-x}$$

$$= \sum_{m=1}^{\infty} m x^{m} + \frac{1}{1-x} = x \sum_{m=1}^{\infty} m x^{m-1} + \frac{1}{1-x} = xH + \frac{1}{1-x}$$

so $H = \frac{1}{(1-x)^2}$. In particular $E[X] = \frac{1}{(1-\frac{6}{7})^2} \cdot \frac{1}{7}$

$$= \frac{1}{(\frac{1}{7})^2} \cdot \frac{1}{7} = \frac{1}{(\frac{1}{7})} = 7.$$