Conditional Probability Lecture 2

In the previous lecture we were causioned not to dismiss information. Take the following example

Ex. Two fair coins are tossed. What is the probability that first coin lands heads up?

Solution: Many students consider the second coin as irrelevant and take the sample space to be $S = \{H, T\}$. Thus $P(H) = \frac{1}{2}$.

Strictly speaking, we are trying to identify all possible universes with the desired property among the set of all potential universes. The second coin is part of every universe to be;

$$S = \{(H,H), (H,T), (T,H), (T,T)\}$$

The probability that first coin comes up heads is

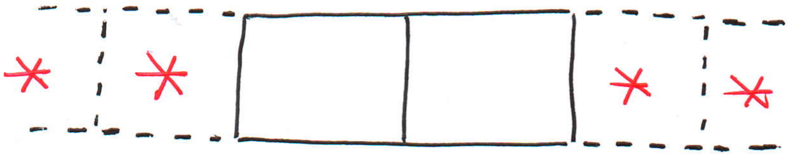
$$P(\{(H,H), (H,T)\}) = \frac{2}{4}$$

Instead of discarding the second coin from our list of universes, we can assume that we know the value on the second coin:

$F = \{(H,H), (T,H)\}$

We can assume we know far more about the universe than we do!

Indeed, we do this all the time. When we toss two fair coins, the actual Kafka protocol must contain information about the entire universe at the moment the coin is tossed. To avoid having to fill out this information, imagine that only the squares containing information about the coins is blank



Ex. 52 cards are dealt equally to 4 players E, W, N, S. If N and S have a total of 8 spades among them, what is the probability that E has 3 of the remaining 5 spades?

Solution: Let E - event $\mathcal{E}$ has 3 spades. Assume that the exact set of 26 cards that belong to $\mathcal{E}$-W is known and that 5 of these 26 cards are spades. Denote this event by F.

$$p(E|F) = \frac{\binom{5}{3}\binom{21}{10}}{\binom{26}{13}} \approx 0.339$$

you may raise the objection that F should have been far less definite. After all, we only know that $\mathcal{E}$-W have some 5 spades among them. We know nothing about the remaining cards. To explain this, we need to introduce a few ideas.

Multiplication Rule

Thm: Let $E_1, \ldots, E_n \subset S$ be events in some sample space S. Then

$$P(E_1 E_2 E_3 \ldots E_n) = p(E_1)\, p(E_2 | E_1)\, p(E_3 | E_1 E_2) \ldots p(E_n | E_1 \ldots E_{n-1})$$

Proof:

$$P(E_1) \cdot \frac{P(E_1 E_2)}{P(E_1)} \cdot \frac{P(E_1 E_2 E_3)}{P(E_1 E_2)} \cdot \frac{P(E_1 E_2 E_3 E_4)}{P(E_1 E_2 E_3)} \cdots \frac{P(E_1 \ldots E_{n-1} E_n)}{P(E_1 \ldots E_{n-1})}$$

$$= P(E_1)\, p(E_2 | E_1)\, P(E_3 | E_1 E_2)\, P(E_4 | E_1 E_2 E_3)\, P(E_n | E_1 \ldots E_{n-1})$$

Ex. Urn contains n balls numbered $1, 2, \ldots, n$ k balls are removed sequentially without replacement in such a way that, at each step, any one of the remaining balls is equally likely to be chosen.

Show that each sequence of k distinct numbers is equally likely.

Solution: There are $\binom{n}{k} k!$ sequences and we wish to show that the probability of each is $\frac{1}{\binom{n}{k} k!}$

Let E_k - event k^{th} chosen ball is ball #k.
Then the probability of the sequence $(1, 2, 3, \ldots, k)$ is $p(E_1 E_2 E_3 \ldots E_k) = p(E_1)p(E_2|E_1) \ldots p(E_k|E_1 \ldots E_{k-1})$

$$= \frac{1}{n} \cdot \frac{1}{n-1} \cdot \ldots \frac{1}{n-k+1} = \frac{1}{\binom{n}{k} k!}$$

where we have used $p(E_j | E_1 \ldots E_{j-1}) = \frac{1}{n-j+1}$

(because ball #j is equally likely to be the one removed among $n-j+1$ remaining balls).

Since there is nothing special about the sequence $(1, 2, 3, \ldots, k)$ we see that all sequences of size k are equally likely.

<u>Ex.</u> (Return to the Matching Problem) If n gentlemen pick out a hat at random, what is the probability that exactly k of them pick their own hat?

Solution: Let E_j - j^{th} gentleman picks own hat.

Desired probability is

$$\binom{n}{k} P(\underbrace{E_1^c E_2^c \dots E_{n-k}^c}_{\text{First } n-k \text{ don't pick own hat}} \underbrace{E_{n-k+1} \dots E_n}_{\text{Last } k \text{ pick own hat}})$$

$$= \binom{n}{k} P(E_{n-k+1} \dots E_n) \, P(E_1^c E_2^c \dots E_{n-k}^c \mid E_{n-k+1} \dots E_n)$$

$$= \binom{n}{k} \frac{(n-k)!}{n!} \cdot \sum_{j=0}^{n-k} (-1)^j \frac{1}{j!}$$

$$= \frac{1}{k!} \sum_{j=0}^{n-k} (-1)^j \frac{1}{j!} \longrightarrow \frac{1}{k!} e^{-1}$$

where we have noticed that $P(E_1^c E_2^c \dots E_{n-k}^c \mid E_{n-k+1} \dots E_n)$ is just the matching problem probability with $n-k$ men.

Bayes's Formula

Recall that $E = ES = E(F \cup F^c) = EF \cup EF^c$

Hence $P(E) = P(\underbrace{EF \cup EF^c}_{\text{disjoint}}) = P(EF) + P(EF^c)$

$$= P(E|F)\,p(F) + p(E|F^c)\,p(F^c)$$

$$= P(E|F)\,p(F) + P(E|F^c)\,(1 - p(F))$$

Ex. Insurance company divides people into two classes: Accident prone and not accident prone. Statistics show that an accident-prone person will have an accident within a year with probability 0.4, whereas this probability is 0.2 for non-accident prone individuals.

If 30% of the population is accident prone, what is the probability that a new policy holder will suffer an accident within a year of getting the policy?

Solution: Let A - event policy holder has accident within a year.

Let I - event policy holder is accident prone (inclined).

We need to compute $P(A)$:

$$P(A) = P(A|I)\,p(I) + p(A|I^c)\,p(I^c)$$

$$= 0.4 \cdot 0.3 + 0.2 \cdot 0.7 = 0.26$$

prob. accident if prone

Prob. accident prone

prob. accident. if not prone

Prob. not accident prone

Ex. New policyholder has an accident within a year of purchasing policy. What is the probability that this individual is accident prone?

Solution: $$P(I|A) = \frac{P(IA)}{P(A)} = \frac{P(A|I)\,p(I)}{P(A)}$$

$$= \frac{0.4 \cdot 0.3}{0.26} = \frac{6}{13} \approx 0.46$$

<u>Ex.</u> A multiple choice question with m possible answers appears on an exam. The probability that student knows the right answer to this question is p. The probability that the student answers correctly by guessing is $\frac{1}{m}$.
What is the probability that student knew the correct answer, given that he answered correctly?

<u>Solution:</u> Let K - event student knows the right answer, and C - event student answers correctly.

$$p(K|C) = \frac{P(KC)}{P(C)} = \frac{P(C|K)P(K)}{P(C|K)P(K)+P(C|K^c)P(K^c)}$$

$$= \frac{1 \cdot P}{1 \cdot P + \frac{1}{m}(1-p)} = \frac{mp}{1+(m-1)p}$$

For instance, suppose that only 10% of the student population know the answer to a given multiple choice question. Suppose 5 possible answers are offered.

Then $P(K|C) = \frac{5 \cdot 0.1}{1 + 4 \cdot 0.1} = \frac{5}{14} \approx 0.36$

Thus, it is more likely that the student guessed.

Ex. An autistic savant wishes to pet a lion in a Russian zoo (where petting lions is allowed for a small fee). He cannot tell a bad lion from a good lion, but knows that the bad lion bites with probability 0.9 and the good lion bites with probability 0.4. If the savant gets mauled by a lion, what is the probability that he went to pet the bad lion?

B

B^c

<u>Solution:</u> Let B - event bad lion gets to be pet

M - event savent gets mauled by lion.

$$P(B|M) = \frac{P(MB)}{P(M)} = \frac{P(M|B)P(B)}{P(M|B)P(B) + P(M|B^c)P(B^c)}$$

where $P(B) = P(B^c) = \frac{1}{2}$ because savent is equally likely to pet either lion.

$$\text{Thus } P(B|M) = \frac{P(M|B)}{P(M|B) + P(M|B^c)} = \frac{0.9}{0.9 + 0.4}$$

$$= \frac{9}{13} \approx 0.69$$

The following is perhaps very relevent in the medical-tests-loving society of rightious hypochondriacs that have taken over all day to day aspects of ordinary existence.

Ex. Vasya is tested for a disease that afflicts 1% of the population. The test result is positive. Assuming this test is 95% reliable, what is the probability Vasya is afflicted with the illness?

Solution: D - event Vasya has the disease.

T - event the test result comes positive.

By 95% reliability we shall mean $P(T|D) = 0.95$ and $P(T^c|D^c) = 0.95$

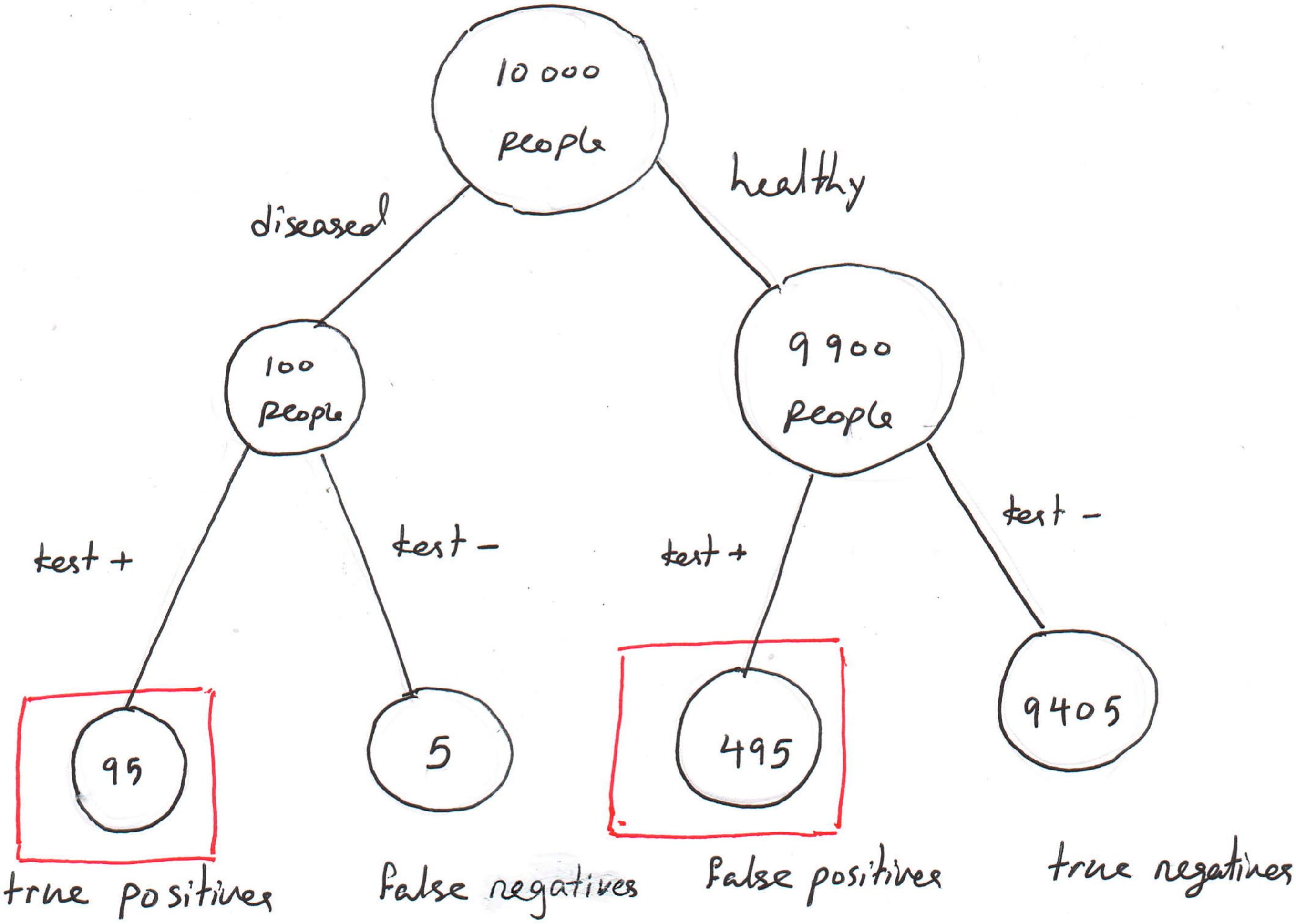
We wish to compute $P(D|T)$.

$$P(D|T) = \frac{P(T|D)\,P(D)}{P(T|D)\,P(D) + P(T|D^c)\,P(D^c)}$$

$$= \frac{0.95 \cdot 0.01}{0.95 \cdot 0.01 + (1-0.95) \cdot 0.99}$$

$$\approx 0.16 \quad \text{or } 16\%$$

Why is this probability so low? Perhaps the following diagram will make the reason a bit more conspicuous.



$$\text{Thus} \quad P(\cdot \mid T) = \frac{\cdot}{\underbrace{95}_{\text{true pos.}} + \underbrace{495}_{\text{false pos.}}}$$

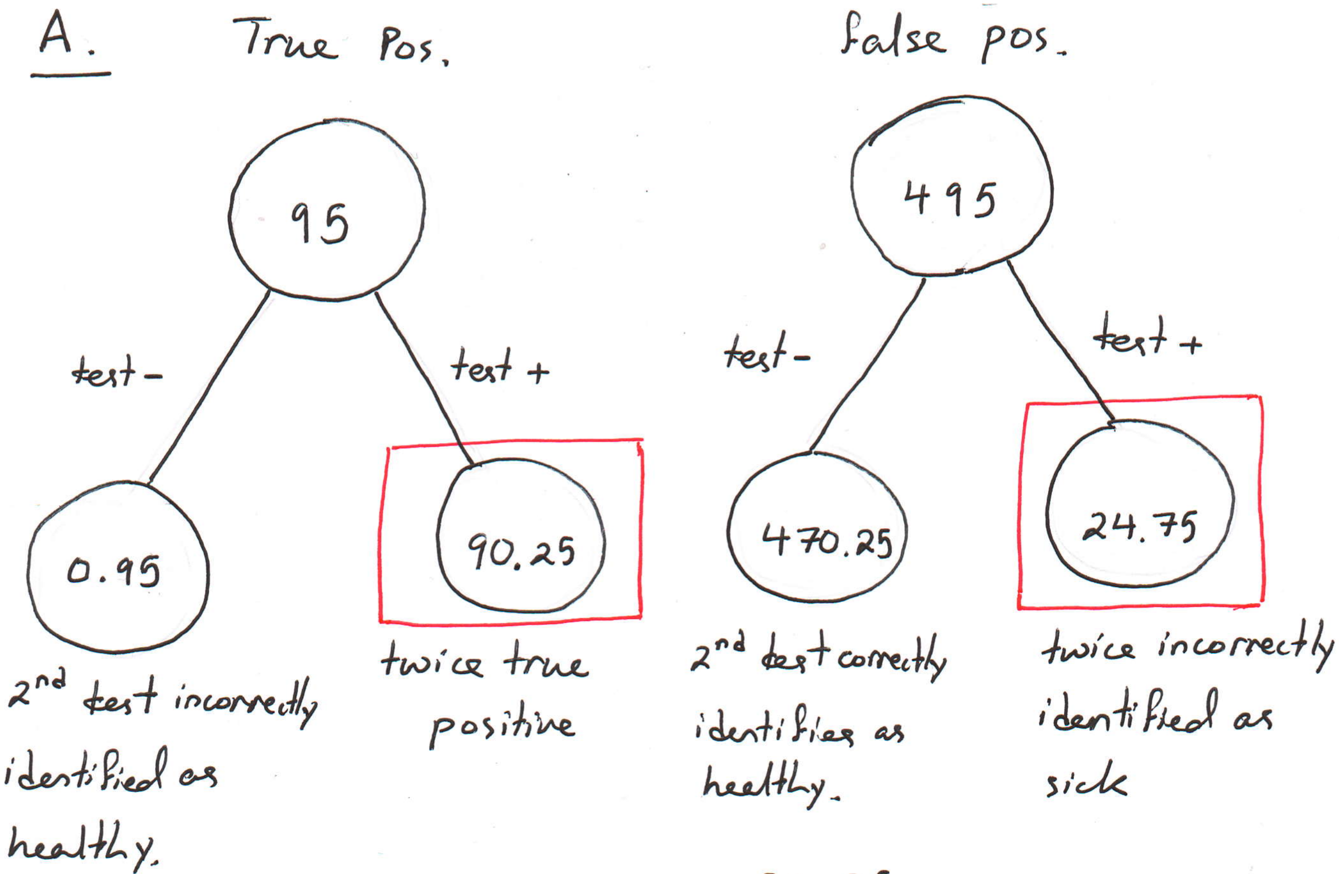
$$\text{so} \quad P(D \mid T) = \frac{95}{95 + 495}.$$

In other words,

there are relatively few sick people living in the universes where test results are positive.

Q. If Vasya is tested again and the result is positive (again). What is the probability that he is afflicted.

A.



$$P(D|T_1 T_2) = \frac{90.25}{90.25 + 24.75} \approx 0.78$$

Ex. In an ongoing criminal investigation, the inspector is 60% certain that the suspect is guilty. New evidence shows that the perpetrator of the crime is left-handed. The suspect is left-handed.
If 20% of the population are left-handed, how certain should the inspector be of the guilt of his suspect?

Solution: Let G - event suspect is guilty; $p(G) = 0.6$. Let L - event suspect is left-handed.

$$p(G|L) = \frac{p(L|G)\,p(G)}{p(L|G)\,p(G) + p(L|G^c)\,p(G^c)}$$

$$= \frac{1 \cdot 0.6}{1 \cdot 0.6 + 0.2 \cdot 0.4} \approx 0.882$$

Generalization of Bayes's Formula

Suppose $F_1, \dots, F_n$ are mutually exclusive events such that $S = \bigcup_{k=1}^{n} F_k$

Then $E = \bigcup_{k=1}^{n} EF_k$. Therefore

$$p(E) = p(EF_1) + p(EF_2) + \dots + P(EF_n)$$

$$= p(E|F_1)P(F_1) + P(E|F_2)P(F_2) + \dots + P(E|F_n)P(F_n)$$

$$= \sum_{k=1}^{n} P(E|F_k)P(F_k)$$

Ex. A plane is missing, and it is presumed that it was equally likely to have gone down in any of 3 possible regions.

Let $1-\beta_k$ denote the probability that plane will be found upon a search of the k^{th} region when the Plane is, in fact, in that region (β_k represents the overlook probability)

What is the probability that plane is in region k if a search in region 1 is unsuccessful?

Solution: Let R_k - event plane in region k and F_1 - event plane was not found in region 1.

$$P(R_1 | F_1) = \frac{P(F_1 | R_1) P(R_1)}{P(F_1|R_1)P(R_1) + P(F_1|R_2)P(R_2) + P(F_1|R_3)P(R_3)}$$

$$= \frac{P(F_1|R_1)}{P(F_1|R_1) + P(F_1|R_2) + P(F_1|R_3)}$$

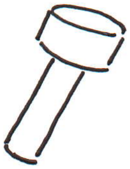
$$= \frac{\beta_1}{\beta_1 + 1 + 1} = \frac{\beta_1}{\beta_1 + 2}.$$

if $k \neq 1$

$$P(R_k | F_1) = \frac{P(F_1|R_k)}{P(F_1|R_1) + P(F_1|R_2) + P(F_1|R_3)} = \frac{1}{\beta_1 + 2}$$

$$\geqslant \frac{\beta_1}{\beta_1 + 2} = P(R_1 | F_1)$$

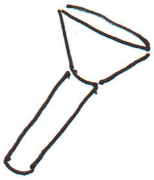
Ex. There are 3 types of flashlights in bin:



Type 1

$P(>100 \text{ hrs}) = 0.7$

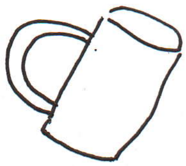
20% of all



Type 2

$P(>100 \text{ hrs}) = 0.4$

30% of all



Type 3

$P(>100 \text{ hrs}) = 0.3$

50% of all

The probability of lasting for over 100 hrs is displayed below each flashlight type.

(a) What is the probability that a randomly chosen flashlight lasts over 100 hrs?

(b) Given that the flashlight lasted over 100 hrs, what is the conditional probability it was a type j flashlight? $(j=1,2,3)$.

Solution: Let A, B, C be type 1, 2, and 3 flashlights and let H be the event chosen flashlight lasts over 100 hrs.

(a) $p(H) = p(HA \cup HB \cup HC) =$

$$= p(H|A)p(A) + p(H|B)p(B) + p(H|C)p(C)$$

$$= 0.7 \cdot 0.2 + 0.4 \cdot 0.3 + 0.3 \cdot 0.5 = 0.41$$

(b) $p(A|H) = \dfrac{p(H|A)\,p(A)}{p(H)} = \dfrac{0.7 \cdot 0.2}{0.41} = \dfrac{14}{41}$

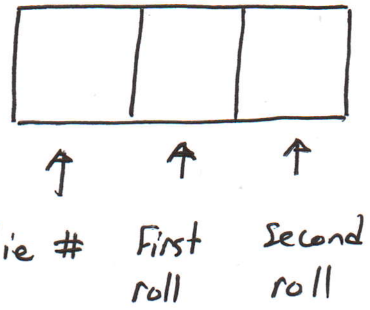
$$p(B|H) = \frac{p(H|B)\,p(B)}{p(H)} = \frac{0.4 \cdot 0.3}{0.41} = \frac{12}{41}$$

$$p(C|H) = \frac{p(H|C)\,p(C)}{p(H)} = \frac{0.3 \cdot 0.5}{0.41} = \frac{15}{41}$$

We can now say a little about assuming greater information than what is given in order to calculate probability.

Ex. One of 4 fair dice will be chosen and tossed twice. (e.g. the dice are $\overset{1}{\text{Blue}}$, $\overset{2}{\text{Red}}$, $\overset{3}{\text{Yellow}}$, $\overset{4}{\text{White}}$). Given that sum on the dice is 7, what is the probability that the numbers 1 & 6 came up?

Solution: The Sample space is of type



Assume that the probability of picking up die k is p_k with $\sum_{k=1}^{4} p_k = 1$

Then $$P(6\&1 \mid sum=7) = \frac{P(6\&1, sum=7)}{P\ (Sum=7)}$$

$$= \frac{\sum_{k=1}^{4} P(6\&1, sum=7, die=k)}{\sum_{k=1}^{4} P(Sum=7, die=k)}$$

$$= \frac{\sum_{k=1}^{4} P(6\&1, sum=7 / die=k)\, P(die=k)}{\sum_{k=1}^{4} P(sum=7 \mid die=k)\, P(die=k)}$$

$$= \frac{\sum_{k=1}^{4} P(6\&1,\ \text{sum}=7 \mid \text{die}=1)\, P(\text{die}=k)}{\sum_{k=1}^{4} P(\text{sum}=7 \mid \text{die}=1)\ P(\text{die}=k)}$$

$$= \frac{\sum_{k=1}^{4} P(6\&1,\ \text{sum}=7 \mid \text{die}=1)\, P_k}{\sum_{k=1}^{4} P(\text{sum}=7 \mid \text{die}=1)\, P_k}$$

$$= P(6\&1 \mid \text{die}=1,\ \text{sum}=7)\ \frac{\sum_{k=1}^{4} P_k}{\sum_{k=1}^{4} P_k}$$

$$= \frac{2}{6} = \frac{1}{3}$$

Thus we may assume that die 1 was chosen and this doesn't affect the calculated probability.