Axioms of Probability Lecture 4

Probability and Infinite Series

In light of Axiom III of probability theory, infinite series are often lurking in the background of many problems.

Ex. (a) A pair of fair dice is rolled until either the sum of 4 or 8 appears for the first time. What is the probability that the game is stopped on the sum 8?

(b) What is the probability in (a) if 4 is replaced with the sum s and 8 is replaced with the sum t, where $s \neq t$ and $2 \leq s, t \leq 12$?

Solution: (a) Let E = event game ends in a sum of 8. Let E_n denote the event that the sum 8 first appears on the n^{th} trial. Clearly

$$E = \bigcup_{n=1}^{\infty} E_n$$

and since $E_n E_m = \emptyset$, we have by axiom III,

$$P(E) = \sum_{n=1}^{\infty} P(E_n).$$

where # ways sum on two dice = 8

$$\left|\{\overset{1}{(2,6)}, \overset{2}{(3,5)}, \overset{3}{(4,4)}, \overset{4}{(5,3)}, \overset{5}{(6,2)}\}\right| = 5$$

and # ways sum on two dice = 4

$$\left|\{\overset{1}{(1,3)}, \overset{2}{(2,2)}, \overset{3}{(3,1)}\}\right| = 3$$

Hence $$P(E_n) = \left(\frac{36-5-3}{36}\right)^{n-1} \frac{5}{36}$$

and $$P(E) = \sum_{n=1}^{\infty} \left(\frac{36-5-3}{36}\right)^{n-1} \frac{5}{36} = \frac{\frac{5}{36}}{1 - \frac{36-5-3}{36}}$$

$$= \frac{5}{36 - (36-5-3)} = \frac{5}{5+3} = \frac{5}{8}.$$

<u>Remark</u>: This problem can be solved more elegantly by noting that the game is equally likely to end in one of the 5+3 outcomes corresponding to the sum = 8 and sum = 3 respectively.

$$P(E) = \frac{\#\{\text{sum}=8\}}{\#\{\text{sum}=8\} + \#\{\text{sum}=4\}}$$

$$= \frac{5}{5+3}.$$

One may, of course, object that there is another possible outcome: The game that never ends.

(b) Let E be the event that t occurs before s. and let E_n be the event that the game is finished on the n^{th} trial with an outcome corresponding to t.

To compute the number of outcomes on a throw of two dice corresponding to t, let x be the value of die #1 and y be the outcome on die #2.

Then

$$x + y = t$$

$$1 \leqslant x \leqslant 6$$

$$1 \leqslant y \leqslant 6$$

Hence $y = t - x$ and satisfies $1 \leqslant t - x \leqslant 6$

or $\quad -1 \geqslant x - t \geqslant -6 \implies t-1 \geqslant x \geqslant t - x$

$$t - 1 \geqslant x \geqslant t - 6$$

$$6 \geqslant x \geqslant 1$$

Means $\min\{t-1, 6\} \geqslant x \geqslant \max\{t-6, 1\}$

Thus $\#\{\text{sum} = t\} = \min\{t-1, 6\} - \max\{t-6, 1\} + 1.$

A similar formula naturally holds for sum $= s$.

For example $\#\{\text{sum} = 10\} = \min\{10-1, 6\} - \max\{10-6, 1\}$

$+1 = 6 - 4 + 1 = 3$

Indeed, $\{sum = 10\} = \{(\overset{1}{4,6}), (\overset{2}{5,5}), (\overset{3}{6,4})\}$

Denoting by $a = \#\{sum = t\}$ and by $b = \#\{sum = s\}$

we expect

$$P(E) = \frac{a}{a+b}.$$

This is also clear from the resulting infinite series.

$$P(E) = \sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} \left(\frac{36-a-b}{36}\right)^{n-1} \frac{a}{36}$$

$$= \frac{\frac{a}{36}}{1 - \frac{36-a-b}{36}}$$

$$= \frac{a}{36-(36-a-b)}$$

$$= \frac{a}{a+b}.$$

Ex. (The Matching Problem) n gentlemen have n hats. If each gentleman takes a hat at random, what is the probability that no one picks his own hat?

Solution: Let E_k = event gentleman k picks his own hat. The desired probability is

$$P\left(\bigcap_{k=1}^{n} E_k^c\right) = 1 - P\left(\bigcup_{k=1}^{n} E_k\right)$$

Now $P\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{i_1 < i_2 < \dots < i_k} P(E_{i_1} E_{i_2} \dots E_{i_k})$

$$= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} P(E_1 E_2 \dots E_k) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{(n-k)!}{n!}$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \frac{\cancel{n!}}{k!\,\cancel{(n-k)!}} \cdot \frac{\cancel{(n-k)!}}{\cancel{n!}} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}$$

Thus

$$P\left(\bigcap_{k=1}^{n} E_k^c\right) = 1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}$$

$$= 1 + \sum_{k=1}^{n} \frac{(-1)^k}{k!}$$

$$= \frac{(-1)^0}{0!} + \sum_{k=1}^{n} \frac{(-1)^k}{k!}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

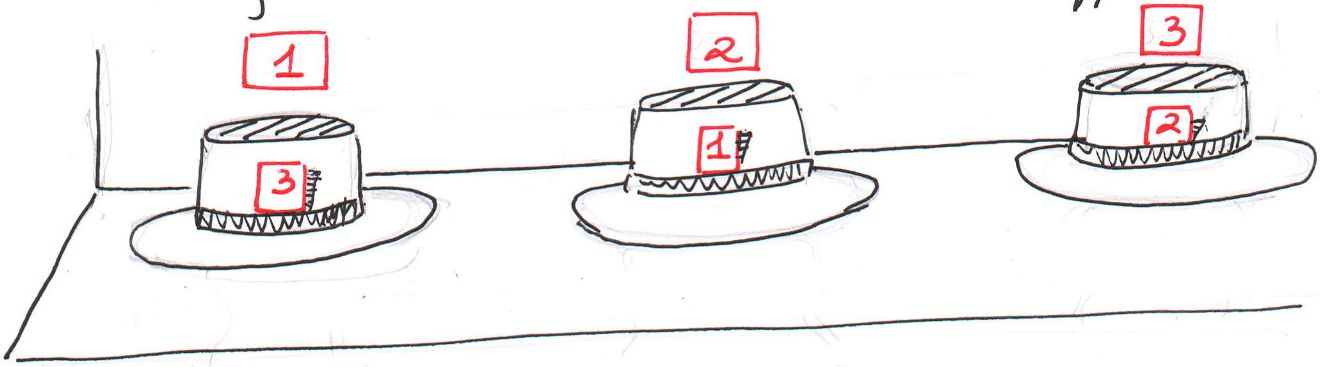
Looks familiar?

Well, notice that $\lim\limits_{n \to \infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$

$= e^{-1}$!

Thus, as the number of gentlemen in the club becomes larger, the probability that no one picks their own hat approaches e^{-1}.

This gives an interesting way to compute the number e: Go to Borough Park, infiltrate a synagogue and randomly swap the hats in the dressing room. Now watch what happens...



$$\frac{\text{Total \# of Pranks}}{\text{\# no one Picks up own hat}} \approx e.$$