Axioms of Probability Lecture 3

Sample Spaces Having Equally likely outcomes

It is often reasonable to assume that all outcomes in the sample space are equally likely to occur.

If $S = \{1, 2, \dots, N\}$

Then $P(1) = P(2) = \dots = P(N) = p$;

$$1 = P(S) = P\left(\bigcup_{k=1}^{N} k\right) = \sum_{k=1}^{N} P(k) = \underbrace{p + p + p + \dots + p}_{N \text{ copies.}}$$

$$= pN$$

Hence $p = \frac{1}{N}$

If $E \subset S$ $\quad p(E) = \dfrac{\#\text{ outcomes in } E}{\#\text{ outcomes in } S}$

Ex. If two dices are rolled, what is the probability that the sum of uptorned faces is 7?

Solution: Let E be the event the sum is 7.

k	$7-k$
Die 1	Die 2

where $1 \leq k \leq 6$

Hence $\#E = 6$

Similarly, $\#S = 6^2$ because the Kafka document for S is

k	j
Die 1	Die 2

$$1 \leq k \leq 6, \quad 1 \leq j \leq 6$$

Thus $P(E) = \dfrac{\#E}{\#S} = \dfrac{6}{6^2} = \dfrac{1}{6}$.

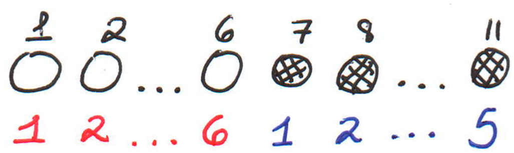
Remark: In many probability problems, the word "random" is being used. Take it to mean "indifferent" or Sample Space in which every outcome is equally likely.

Ex. 3 balls to be randomly drawn from a bowl containing 6 white and 5 black balls. What is the probability that one ball is white and the other 2 black?

Solution: Probability problems of this sort are combinatorial problems × 2. Just be sure to count the types of objects for the numerator and denominator. That is, keep in mind what is your sample space.

Assumption 1

- All balls are distinguishable
- Balls are drawn in order

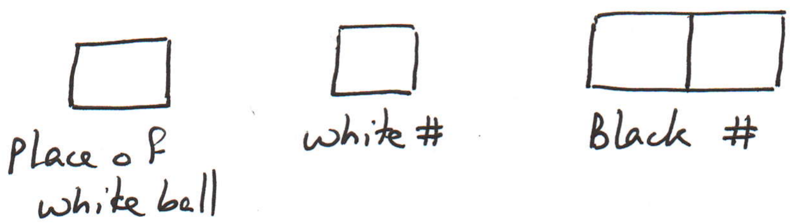


1, 2, ..., 11 – population ID

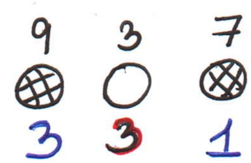
1 2 ... 6 1, 2 ... – color ID.

To construct a kafka document for the numerator, imagine each ball is assigned two IDs. One number designates the ball's id in the general population while the other number gives the ball a color id.

Let E = event 1 ball is white, 2 balls are black



For example, if the outcome is

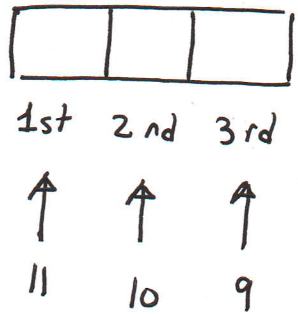


we write the code $\boxed{2}, \boxed{3} \quad \boxed{3}\boxed{1}$

$\uparrow$ 3 pos. $\quad \uparrow$ 6 pos. $\quad \uparrow$ 5 $\quad \uparrow$ 4

Hence $\#E = 3 \cdot 6 \cdot 5 \cdot 4 = 3! \binom{6}{1} \binom{5}{2}$

The kafka form for S can be taken to be



Thus $P(E) =$

$$\frac{\# E}{\# S} = \frac{3 \cdot 6 \cdot 5 \cdot 4}{11 \cdot 10 \cdot 9} = \frac{4}{11}$$

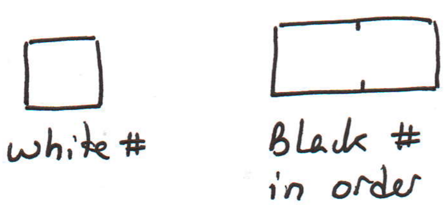
Notice that $\dfrac{\# E}{\# S} = \dfrac{3! \binom{6}{1}\binom{5}{2}}{3! \binom{11}{3}} =$

$$= \frac{\binom{6}{1}\binom{5}{2}}{\binom{11}{3}}$$

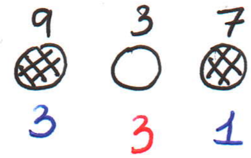
Assumption 2

- All balls are distinguishable
- 3 balls are sampled simultaneously.

Kafka form for E



For example, the outcome



will be registered as

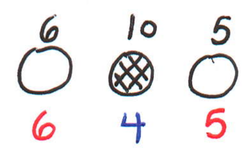
$$\boxed{3}\ \ \boxed{1 \mid 3}$$

$\binom{6}{1}$ pos. $\binom{5}{2}$ pos.

$$\# E = \binom{6}{1} \cdot \binom{5}{2}.$$

The kafka form for the denominator could then be

$$\boxed{5 \mid 6 \mid 10} \longleftarrow$$



selected pop.
in order

$\binom{11}{3}$ pos.

Thus $$P(E) = \frac{\binom{6}{1}\binom{5}{2}}{\binom{11}{3}} = \frac{4}{11}$$

Notice that even though we worked with two different sample spaces in assumptions 1 and 2, the end result ended-up to be the same.

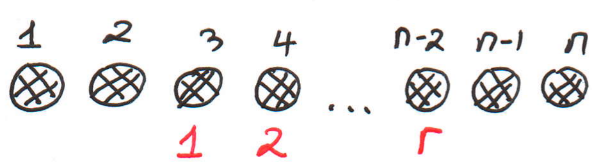
This is because a quotient of two combinatorial problems can be the same as the quotient of two different combinatorial problems.

<u>Observation:</u> Sequential random sampling generates the same probabilities as simultaneous random sampling.

Let n be the number of balls in the bowl.

Let r be the number that will be sampled.

Designate by $1-r$ a particular batch of balls.



Then the simultaneous sampling probability of $1-r$ is

$$P_{sim} = \frac{1}{\binom{n}{r}}$$

whereas the sequential sampling probability is

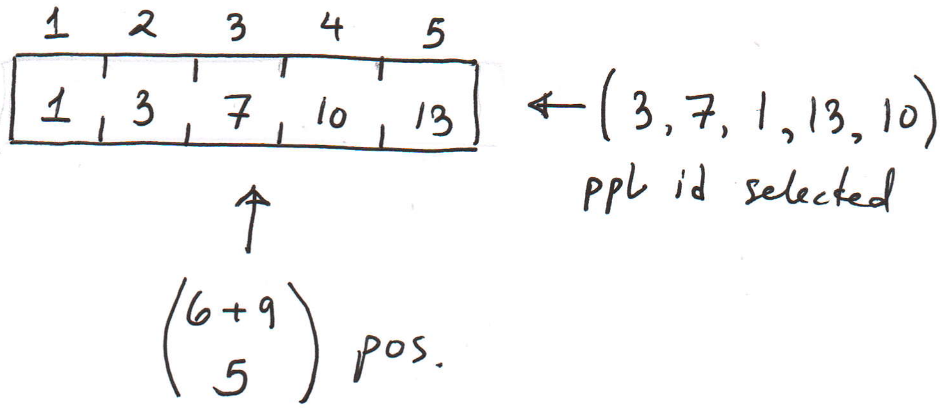
$$P_{sec} = \frac{r!}{r!\binom{n}{r}} = P_{sim}$$

Ex. A Committe of 5 is selected from a group of 6 men and 9 women. If selection is made randomly what is the probability of 3 men and 2 women?

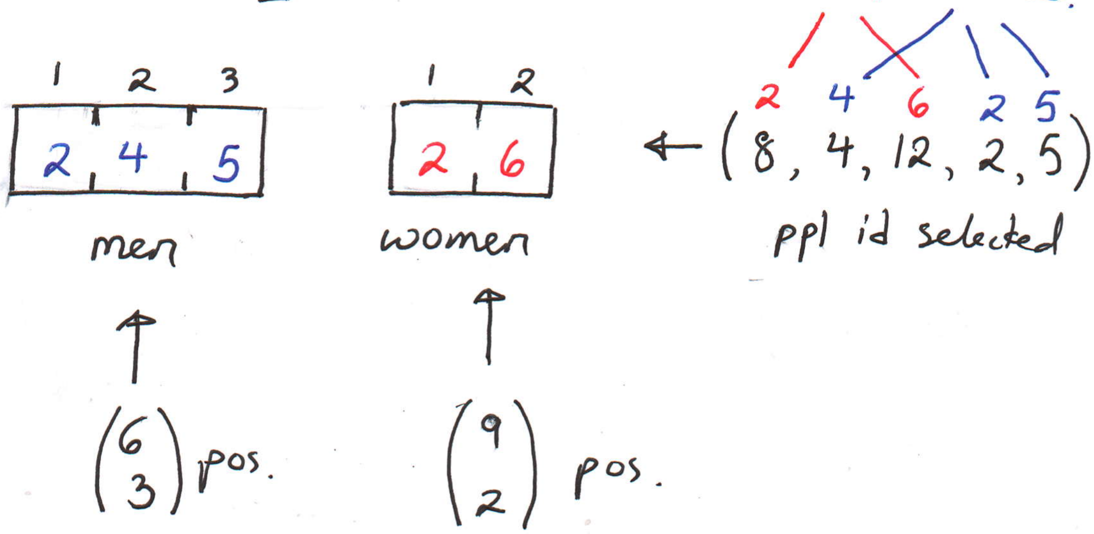
Solution: Assign 6+9 people ids and 6 male, 9 female identification cards to the objects.

Sample Space: 5 selections grouped by person rank.

(9)



Desired Event E :



Hence the probability is

$$P = \frac{\binom{6}{3}\binom{9}{2}}{\binom{6+9}{5}}$$

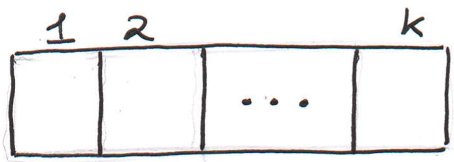
Making a clever choice for the sample space can drastically reduce your calculations.

<u>Ex.</u> There are n balls in the urn. One ball is special. If k balls are randomly withdrawn, what is the probability that the special ball is chosen?

<u>Solution:</u> Assign ids $1-n$ to the balls n = id of spec.

(1) Selection is carried out sequentially.

S :



$k!\binom{n}{k}$ pos.

$$E:\quad \boxed{2}\qquad \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{k-1}\\ \hline 1 & 2 & \cdots & k-1 \\ \hline \end{array} \longleftarrow \underset{1\quad 2\quad 3\quad \dots\quad k}{(1, n, 2, \dots, k-1)}$$

moment of ball n select. $\uparrow$ k pos.

relative selection order of non-spec. balls $\uparrow$ $\binom{n-1}{k-1}(k-1)!$ pos.

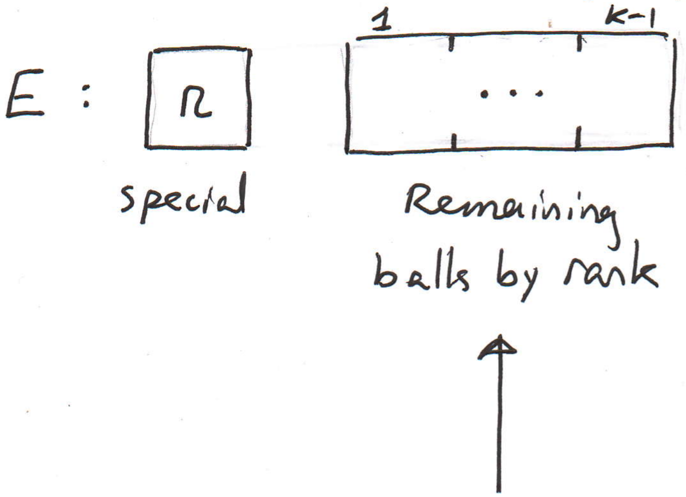
Hence
$$P = \frac{k\binom{n-1}{k-1}(k-1)!}{k!\binom{n}{k}} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}}$$

$$= \frac{n\binom{n-1}{k-1}}{n\binom{n}{k}} = \frac{k\cancel{\binom{n}{k}}}{n\cancel{\binom{n}{k}}} = \frac{k}{n}.$$

(2) Selection is carried out simultaneously.

$$S:\quad \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{k}\\ \hline & & \cdots & \\ \hline \end{array}$$

List by rank $\uparrow$ $\binom{n}{k}$ pos.

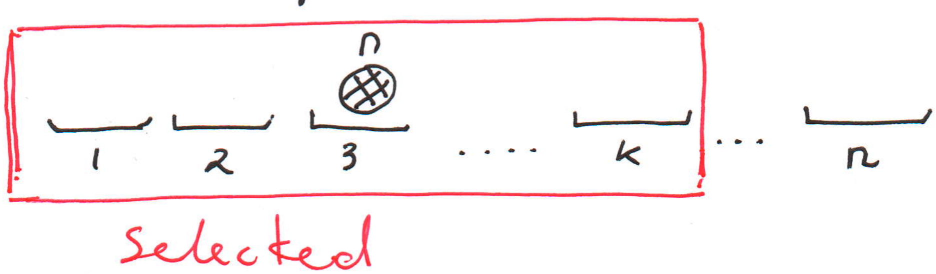


$\binom{n-1}{k-1}$ pos.

Hence $P = \dfrac{\binom{n-1}{k-1}}{\binom{n}{k}} = \dfrac{k}{n}$

(3) Clever solution. Imagine the balls are randomly placed into a Pez dispensor. The first k "candies" are picked.

S : Place of special ball in the dispensor



S : $\boxed{3}$

place of spec. ball in Rez

$\uparrow$

n pos.

E : $\boxed{}$ $\Longrightarrow P = \dfrac{\# E}{\# S} = \dfrac{k}{n}$.

Place of selected

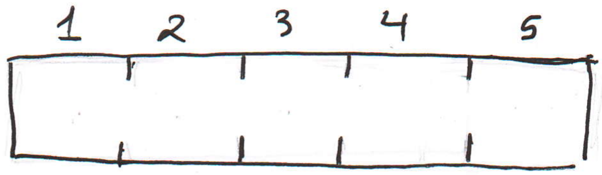
$\uparrow$

k pos.

<u>Ex.</u> Poker hand = 5 cards. If cards have distinct consecutive values and are not all of the same suit, we say that the hand is a straight. What is the probability that one is dealt a straight? Assume all possible hands are equally likely.

<u>Solution:</u> Assign every card a population id # (1-52).

S :



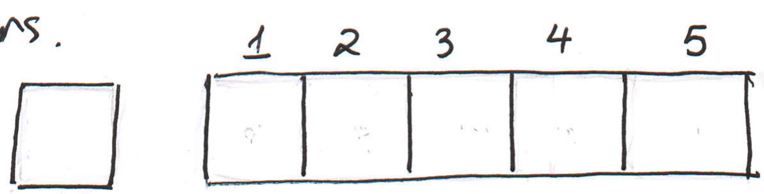
Simultaneously select 5
Order by pop. rank

$$\binom{52}{5}$$

We can think of each card as having also a "front" number and a "back" number (e.g. ace of hearts (1, 1))

There are 13 front numbers and 4 back numbers.

E :



First conseq. #

Back (Suit) #
Listed by front ranking.

$13 - 5 + 1$

$4^5 - 4$

same suit 1
⋮
same suit 4

Thus $P = \dfrac{9 \cdot (4^5 - 4)}{\binom{52}{5}} \approx 0.0035$

<u>Ex.</u> If it is assumed that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of getting

(a) One pair (a, a, b, c, d) $a \neq b \neq c$

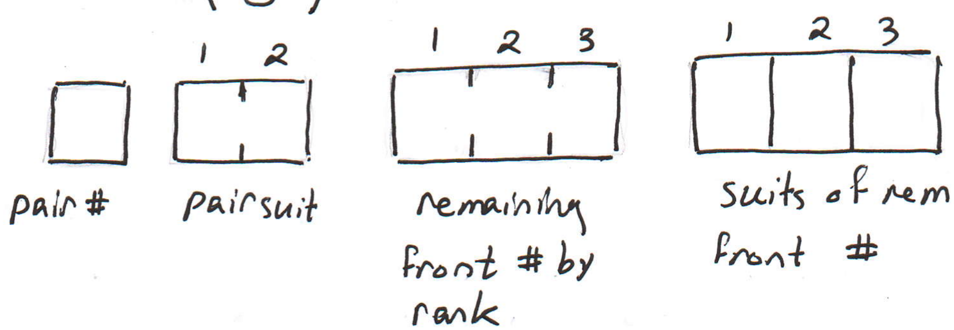
(b) Two Pairs (a, a, b, b, c) $a \neq b \neq c$

<u>Solution:</u>

S : Simultaneous 5 card selections

$$\#S = \binom{52}{5}$$

(a) E :



Thus $\#E = \binom{13}{1}\binom{4}{2}\binom{12}{3}4^3$

Hence $P = \dfrac{\binom{13}{1}\binom{4}{2}\binom{12}{3}4^3}{\binom{52}{5}} \approx 0.4226$

(b) E :

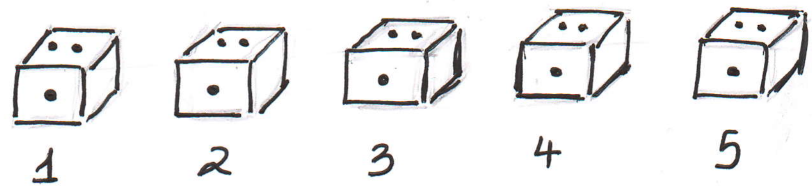
values on pairs	suits for pairs: small	large	remaining card
$\binom{13}{2}$	$\binom{4}{2}$	$\binom{4}{2}$	$\binom{52-8}{1}$

Hence

$$P = \frac{\binom{13}{2}\binom{4}{2}^2 \cdot 44}{\binom{52}{5}} \approx 0.0475$$

<u>Ex.</u> Poker dice is played by simultaneously rolling 5 dice. What is the probability of two pairs?

Solution:



Regard each die as unique.

The sample space may be described by the Kafka protocol

1	2	3	4	5

$\uparrow$

6^5 pos.

The protocol for two pairs could be

2	4

pair values

$\uparrow$

$\binom{6}{2}$ pos.

3

single value

$\uparrow$

$\binom{4}{1}$ pos.

1	3	4	5	2

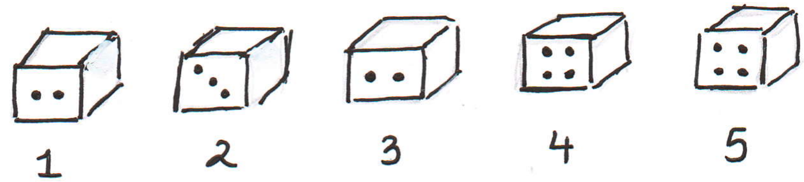
Dice displaying values

$\uparrow$

$$\binom{5}{2 \; 2 \; 1}$$

For example, the code 24 3 13452

means



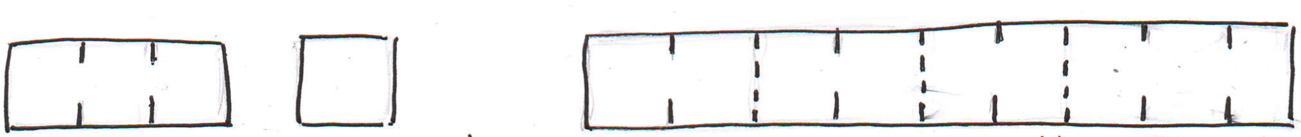
Hence

$$p = \frac{\binom{6}{2}\binom{4}{1}\binom{5}{2\ 2\ 1}}{6^5} \approx 0.23\overline{148}$$

__Ex.__ 9 dice are simultaneously rolled. What is the probability of getting 3 pairs and a tripple?

$(a, a, b, b, c, c, d, d, d)$

__Solution:__ The protocol for denominator is clear. The protocol for the desired outcomes could be



on pairs $\uparrow$ $\binom{6}{3}$ pos.

on tripple $\uparrow$ $\binom{3}{1}$

Dice carrying these values $\uparrow$ $\binom{9}{2\ 2\ 2\ 3}$

Thus

$$p = \frac{\binom{6}{3}\binom{3}{1}\binom{9}{2\,2\,2\,3}}{6^9}$$

<u>Ex.</u> In a game of Bridge, 52 cards are dealt equally to 4 players. What is the probability that

(a) One of the players receives all 13 spades?

(b) Each player receives 1 ace?

<u>Solution:</u>

(a) Let R_k be the event that player k receives all the spades. We are interested in

$$p\left(\bigcup_{k=1}^{4} R_k\right) = 4\,p(R_1) \text{ where}$$

$$P(R_1) = \binom{39}{13\;13\;13} \Big/ \binom{52}{13\;13\;13\;13} = \frac{\left(\frac{39!}{(13!)^3}\right)}{\left(\frac{52!}{(13!)^4}\right)}$$

$$= \frac{39! \cdot 13!}{52!}$$

and the desired probability is therefore

$$P\left(\bigcup_{k=1}^{4} R_k\right) = \frac{4 \cdot 39! \cdot 13!}{52!} \approx 6.3 \times 10^{-12}$$

(b) The desired probability is

$$\frac{4! \binom{48}{12\ \ 12\ \ 12\ \ 12}}{\binom{52}{13\ \ 13\ \ 13\ \ 13}} = \frac{4!\ 13^4\ (48)!}{(52)!}$$

$$= \frac{4! \cdot 13^4}{52 \cdot 51 \cdot 50 \cdot 49} \approx 0.105$$

<u>Comprehension Check:</u> 6 cards are sampled randomly from a regular deck of 52 cards. Calculate the probability of being dealt 3 pairs?

Solution: Let S be the collection of all subsets of size 6 from the set of 52 cards.

The protocol for reporting the outcomes is clear.

3	7	15	17	23	51

Selected cards by rank

$\{3, 15, 17, 51, 23, 7\}$

Hence $\#S = \binom{52}{6}$

The protocol for 3 pairs may be as follows:

2	4	11

pair # smallest – largest

$\binom{13}{3}$ pos.

1	3

suits for smallest pair

2	4

suits for middle p.

1	4

suits for largest p.

$\binom{4}{2}\binom{4}{2}\binom{4}{2}$ pos.

Thus

$$P = \frac{\binom{13}{3}\binom{4}{2}\binom{4}{2}\binom{4}{2}}{\binom{52}{6}} \approx 0.003$$

<u>Ex.</u> (The birthday problem) If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day?

<u>Solution:</u> If no two people share a b-day, there are $365 \cdot 364 \cdot \ldots \cdot (365-n+1)$ birthday assignments for the n individuals. Thus

P(no two people have the same b-day) =

$$= \frac{365 \cdot 364 \cdot \ldots \cdot (365-n+1)}{365^n}$$

when $n \geq 23$, this probability sinks below $\frac{1}{2}$

you can see that this is true by experimenting with the value n, but this is tedious. Instead, imagine that we test for b-day matches by pairing any two people to see if they might have the same birthday.

Let E_{kj} be the event that person k and person j share the same b-day $(k < j)$

Then $P(E_{kj}) = \dfrac{365}{365^2} = \dfrac{1}{365}$

Hence $P(E_{kj}^c) = \left(1 - \dfrac{1}{365}\right)$

Clearly there are $\binom{n}{2} = \dfrac{n(n-1)}{2}$ people combinations. The probability of no matches is therefore approximately $\left(1 - \dfrac{1}{365}\right)^{\frac{n(n-1)}{2}}$

$$\underset{1}{\underline{\text{No}}} \quad \underset{2}{\underline{\text{No}}} \quad \underset{3}{\underline{\text{No}}} \quad \cdots \quad \underset{\binom{n}{2}}{\underline{\text{No}}}$$

$$\left(1 - \frac{1}{365}\right) \left(1 - \frac{1}{365}\right) \cdots \left(1 - \frac{1}{365}\right)$$

We will discuss multiplication of probability in the next section.

$$P(\text{No matches}) \approx \left(1-\frac{1}{365}\right)^{\frac{n(n-1)}{2}} = \left[\left(1-\frac{1}{365}\right)^{365}\right]^{\frac{n(n-1)}{2\cdot 365}}$$

$$\approx \left(e^{-1}\right)^{\frac{n(n-1)}{720}}$$

This probability is less than p if

$$\boxed{e^{-\frac{n(n-1)}{720}} < p}$$

or

$$\frac{-n(n-1)}{720} < \ln p$$

$$n^2 - n + 720\ln p > 0$$

$$n \geq \frac{1+\sqrt{1-4\cdot 720\ln p}}{2}$$

For $p = \frac{1}{2}$, $n \geq 22.999$ so $n = 23$

For $p = \frac{1}{10}$, $n \geq 41.5017$ so $n = 42$.

Hence, with 42 or more people, the probability that at least two of them share a common b-day is at least 0.9.

<u>Ex.</u> A deck of 52 cards shuffled. Cards are turned up one at a time until the first ace appears. Is the next card (the one after the first ace) more likely to be the ace of spades or the two of clubs?

<u>Solution:</u> Assume all card orderings are equally likely. We can work with the following sample space:

S = all possible linear orderings of 52 cards

$\#S = 52!$

Let A_s = event ace of spades after first ace

and 2_c = event 2 of clubs after first ace.

To count $\# A_S$, take the ace of spades out and note that there are 51! orderings of the remaining cards. each ordering determines where ace of spades must be placed.

$$P(A_S) = \frac{51!}{52!} = \frac{1}{52}$$

Similarly

$$P(2_c) = \frac{51!}{52!} = \frac{1}{52}.$$

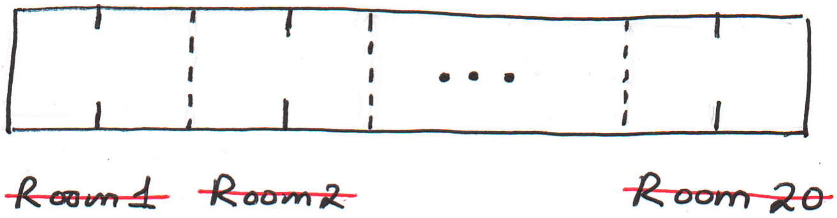
The probability is the same!

<u>Ex.</u> 20 offencive and 20 defensive players in a football team. Players are randomly paired in groups of 2 as roommates.

(a) What is the probability that there are no offensive-defensive roommate pairs?

(b) What is the probability that there are $2k$ offensive-defensive roommate pairs, $k = 1, 2, \dots, 10$?

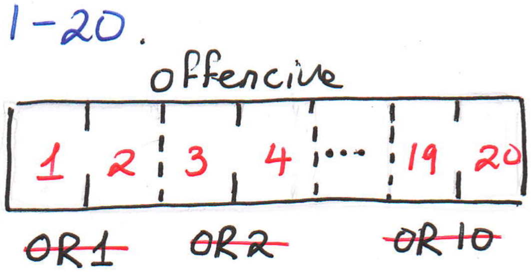
Solution: The kafka protocol for the sample space could be as follows:

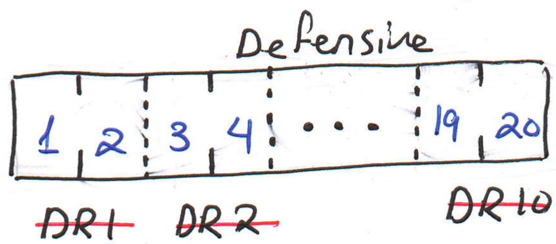


$$\frac{1}{20!}\binom{40}{2\ 2\ \ldots\ 2} = \frac{40!}{20!\ 2^{20}}$$

where the players were assigned population numbers 1-40. We divide by 20!, because the categories Room 1, Room 2, ..., Room 20 are not distinguishable

(a) Label the offencive players with red numbers 1-20. Label the defensive players with blue numbers 1-20.





Thus there are

$$\frac{\binom{20}{2\ 2\ \dots\ 2}\binom{20}{2\ 2\ \dots\ 2}}{10!\quad 10!} = \frac{(20!)^2}{(10!)^2\, 2^{20}}$$

offensive and defensive pairs grouped seperately.

Hence the desired probability is

$$\frac{\left(\dfrac{(20!)^2}{(10!)^2 \cdot 2^{20}}\right)}{\left(\dfrac{40!}{20!\, 2^{20}}\right)} = \frac{(20!)^3}{(10!)^2\, 40!} \approx 1.34 \times 10^{-6}$$

(b) To have $2k$ pairs, pick $2k$ offensive and $2k$ defensive players. There are $(2k)!$ ways to assign them into offensive-defensive pairs. The rest is as in part (a).

$$\frac{\binom{20}{2k}\binom{20}{2k}(2k)!\ \dfrac{(20-2k)!}{2^{10-k}(10-k)!} \cdot \dfrac{(20-2k)!}{2^{10-k}(10-k)!}}{\left(\dfrac{40!}{2^{20}\ 20!}\right)}$$