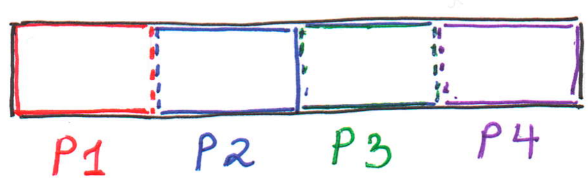
Combinatorial Analysis Lecture 7

Farewell to Combinatorics

1) 52 cards to be divided equally among 4 players. How many possible outcomes?

Solution:

Label cards 1-52 and construct Kafka protocol



where each colored box contains exactly 13 numbers arranged in increasing order within the box.

Clearly the number of outcomes is $\binom{52}{13\ 13\ 13\ 13} =$

$$= \frac{52!}{(13!)^4}$$

2) 52 cards divided into 4 equal piles.

How many outcomes?

Solution:

This time, we can uncouple the colored boxes of the previous problem and rearange them in 4! different ways. Thus there are

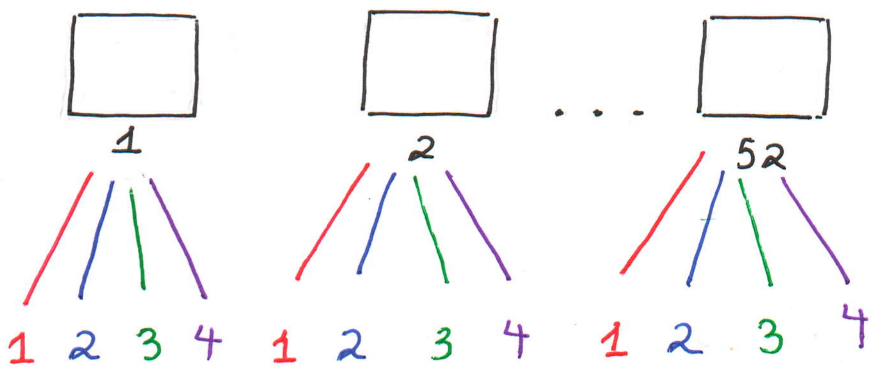
$$\frac{1}{4!}\binom{52}{13\ 13\ 13\ 13} = \frac{52!}{4!\,(13!)^4}$$ possibilities.

3) 52 cards are divided among 4 players, such that each player may receive any number of cards (from 0 to 52).

How many outcomes?

Solution:

The Kafka document for this experiment would consist of 52 boxes - One for each card.
Each box will be assigned a number 1-4.
This represents the player that will receive the card.



This accounts for 4^{52} possibilities (by the generalized principle of counting).

4) 52 cards divided into 4 piles. The piles are not necessarily of equal size.
How many partitions are possible?

Solution:

I could not figure out a way to solve this problem without using recursion.

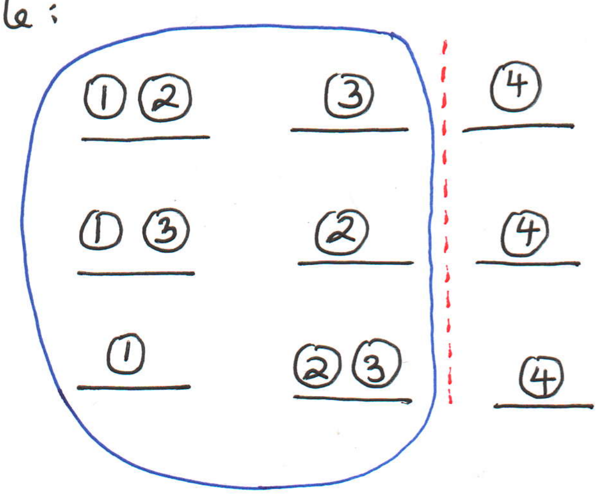
Define $N_k(n)$ – # of partitions of n distinct objects into k piles.

Clearly $N_n(n) = 1$ and $N_1(n) = 1$ for all $n \in \mathbb{N}$. Also $N_k(n) = 0$ if $k > n$.

Focusing on the n^{th} object, we get

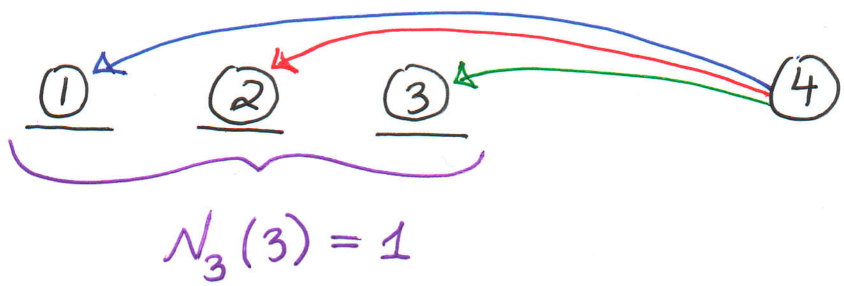
$$N_k(n) = \underbrace{N_{k-1}(n-1)}_{\text{object } n \text{ is alone in the pile}} + \underbrace{k\,N_k(n-1)}_{\text{object } n \text{ is not alone in the pile.}}$$

For example, to calculate $N_3(4)$, note that there are $N_2(3) = 3$ outcomes in which #4 is alone in the pile:



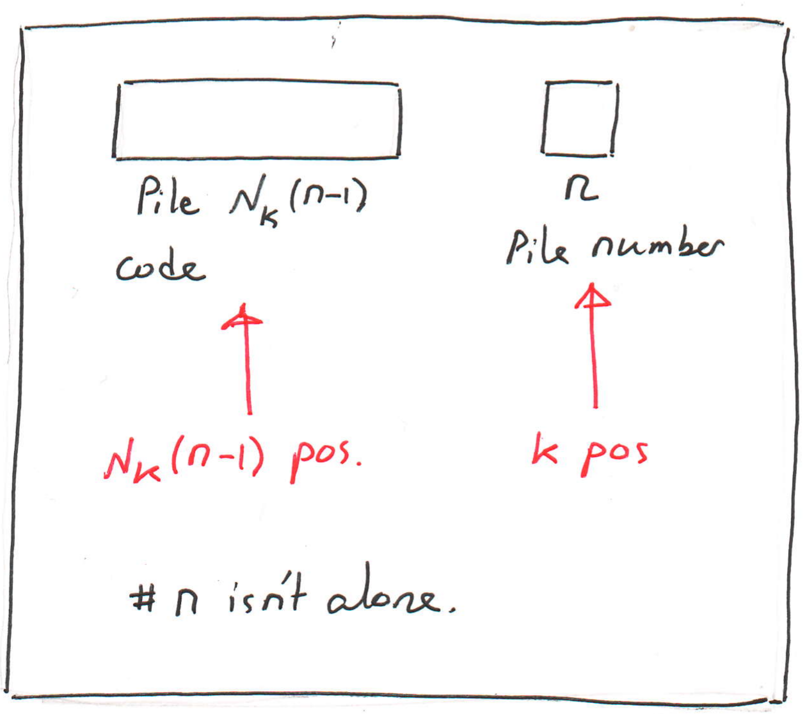
$N_2(3) = 3$

On the other hand, if #4 isn't alone, removing it leaves us with 3 elements distributed among 3 piles:



We multiply $N_3(3)$ by the number of piles 3, because #4 may belong to any one of the 3 piles.

The general Kafka documents for the situation, where #n isn't alone may look like this:



$N_k(n-1) \cdot k$ different doc.

We can check that the recursion works:

$$N_3(4) = N_2(3) + 3N_3(3)$$
$$= N_2(3) + 3$$
$$= N_1(2) + 2N_2(2) + 3$$
$$= 1 + 2 + 3 = 6$$

And we have already seen that

$N_3(4)$ = # piles where ④ alone

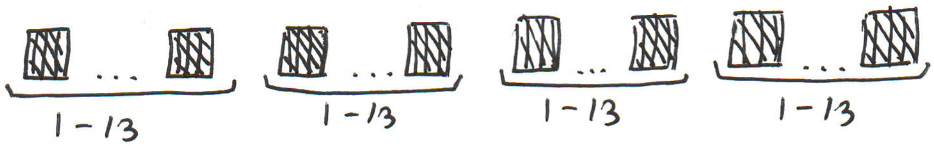
+ # piles where ④ isn't alone

= $3 + 3 = 6$.

For $n = 52$ cards and $k = 4$ piles, we get $N_4(52)$.

5) 52 cards are dipped in black ink (they are no longer distinguishable).
How many ways of dividing them into 4 equal piles?

Solution:



Only 1 way!

6) 52 black cards (not distinguishable) are divided into 4 (possibly unequal) piles among 4 players. How many outcomes are possible?

Solution:

Let X_k - # number of cards assigned to player k, $X_k \geq 1$.

We want to know the number of solutions to the equation $X_1 + X_2 + X_3 + X_4 = 52$.

Clearly there are $\binom{51}{3}$ solutions.

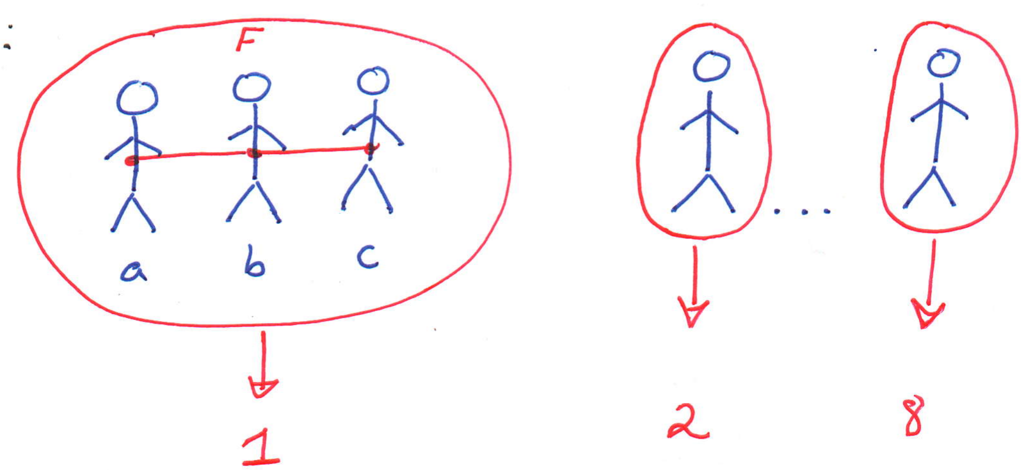
7) 10 Children come to a party. 3 of them are very good friends and will only sit together.

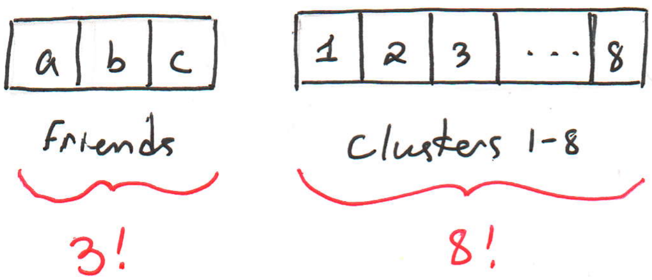
(a) How many ways of sitting them in a row?

(b) How many ways of sitting them on a merry-go-round?

Solution:

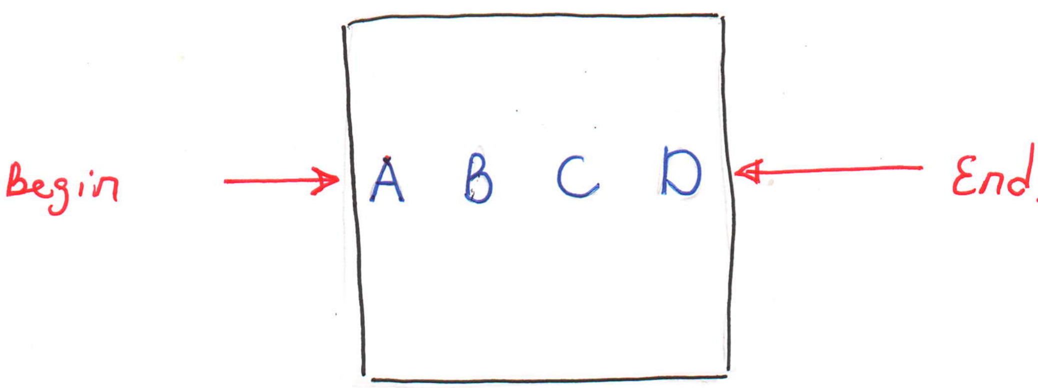
(a) Construct Kafka protocol, in which you describe the experiment of merging the 3 friends into a human centipede, label it as object 1 and count the number of ways of coupling it with the remaining 7 objects:





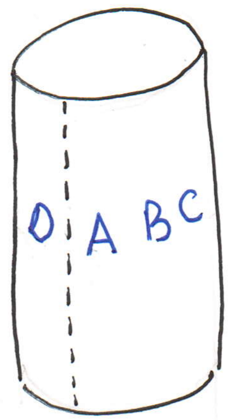
Thus the desired outcome is $3! \cdot 8!$

(b) <u>Idea:</u> Reading a word on a piece of paper is unambiguous. Clear beginning and end:



This is linear ordering.

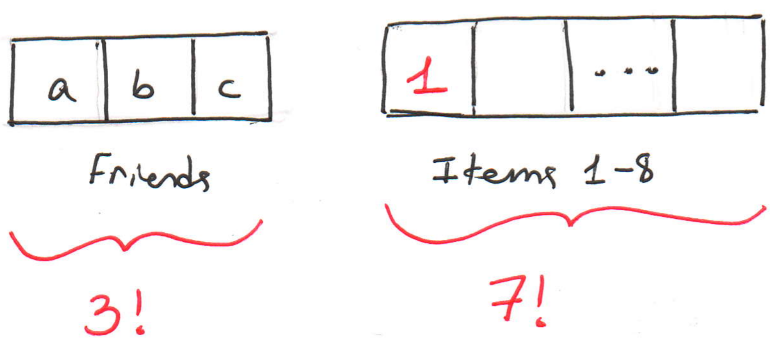
If we fuse the vertical edges seamlessly, we obtain a cylinder with no beginning or end.



The words ABCD, BCDA, CDAB, and DABC are all indicative of the same pattern.

We may, however, decide to arrange the beginning and end line to be drawn between any two letters!

Assume that the merry-go-round rotates clockwise when viewed from above. We will note the sequence of items starting with cluster 1 (the 3 friends):



In the second box, first number is always 1.

Thus there are $3! \cdot 7!$ possibilities.

8) n Children total.

n_1 - from Summer Camp 1

n_2 - from Summer Camp 2

$\vdots$

n_k - from Summer Camp k

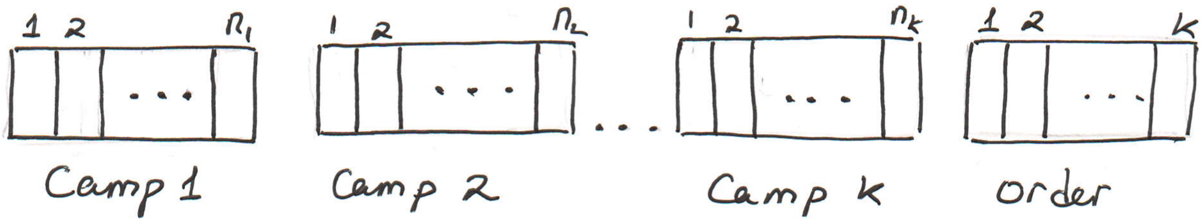
$n_1 + n_2 + \ldots + n_k = n$

(a) How many ways to sit them in a row if children from the same camp must sit together?

(b) How many ways to sit them on a merry-go-round if children from the same camp must sit together?

Solution:

(a) We arrange the children in accordance with the Kafka protocol, which describes how the camp clusters are to be formed and how the camps will be coupled:



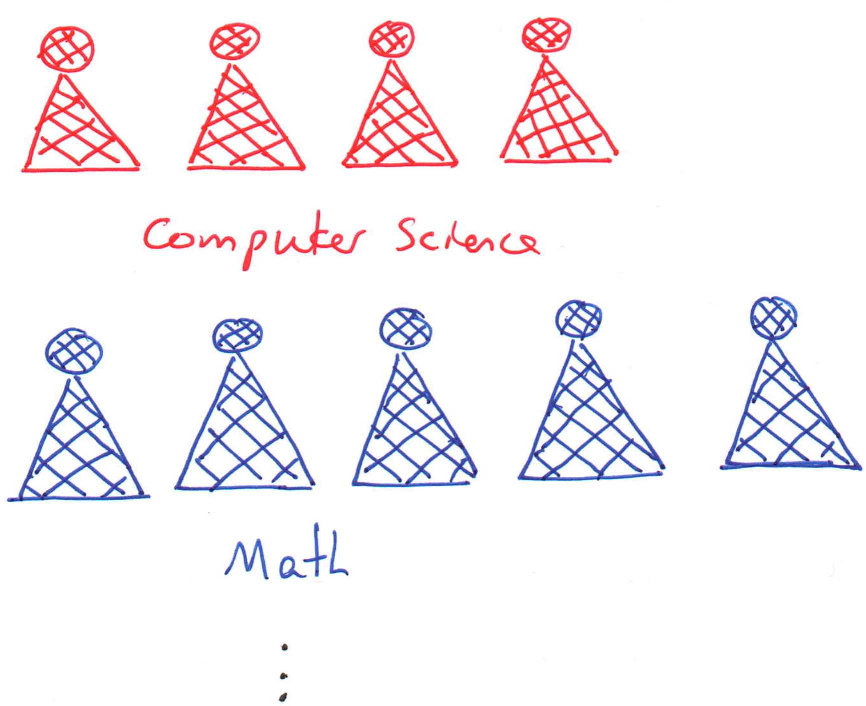
Clearly there are $n_1! \cdot n_2! \cdot \ldots \cdot n_k! \cdot k!$ ways of filling out this form.

(b) We may use the same document as above but begin with the code 1 in the "Order" box (why?)

Thus there are $n_1! \cdot n_2! \cdot \ldots \cdot n_k! \, (k-1)!$ distinct orderings on a merry-go-round.

9) 4 Computer science, 5 math, and 10 sociology majors are taking a 6hr probability exam. As you know, 2 cannot tell students from the same category. 2 can only tell the species of the student and the hour when it leaves. How many possible exit patterns can 2 see?

Solution:



We see 3 equations:

$$x_1 + \dots + x_6 = 4$$

$$y_1 + \dots + y_6 = 5$$

$$z_1 + \dots + z_6 = 10$$

where x_k, y_k, z_k are the number of computer science, math, and sociology students respectively. that leave the examination room at hour k

So the desired number of outcomes is

$$\binom{4+5}{5}\binom{5+5}{5}\binom{10+5}{5}$$

$$= \binom{9}{5}\binom{10}{5}\binom{15}{5}$$