Combinatorial Analysis Lecture 6

Distributing Identical Objects among Non-identical Destinies

Ex. 5 toilet paper rolls to be divided among 6 hypochondriacs. How many partitions are possible?

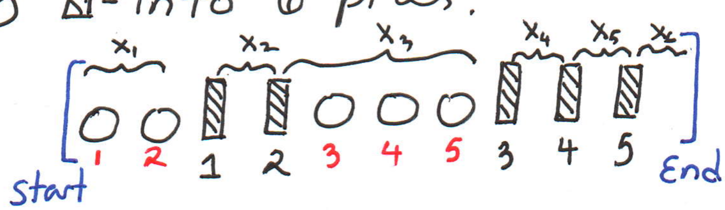
Solution: We want to know the number of integer solutions to the equation

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 = 5$$

where X_k – # rolls obtained by hypochondriac # k.

Idea: Groceries are divided by ▨! One ▨ partitions the rolls into two piles. Two ▨ – into 3 piles... 5 ▨ – into 6 piles.

For example



means $X_1 = 2,\ X_2 = 0,\ X_3 = 3,\ X_4 = X_5 = X_6 = 0$

That is, we can encode each partition by alotting $5+5$ spaces (spaces 1-10) and then choosing which spaces will be occupied by the rolls and the bars respectively

Hence the answer is

$$\binom{5+5}{\underbrace{5}_{\text{spaces for toilet rolls}}\ \underbrace{5}_{\text{spaces for bars}}} = \binom{10}{5}$$

<u>Ex.</u> n toilet paper rolls to be divided among m hypochondriacs. How many partitions are possible?

<u>Solution:</u> n indistinguishable objects (toilet rolls) to be assigned among m distinguishable destinies.

$$X_1 + X_2 + \dots + X_m = n$$

where X_k – # rolls to hypochondriac k.

Clearly $m-1$ bars are needed to create m partitions.

Thus,

Total number of space slots $= n+m-1$

slots assigned to toilet rolls $= n$

Slots assigned to bars $= m-1$

Hence the multinomial coefficient

$$\binom{n+m-1}{n \quad m-1} = \binom{n+m-1}{n}$$

$$= \binom{n+m-1}{m-1}$$

Ex. Do you remember "The Little Match Girl" by Hans Christian Andersen? The poor girl would light up a match to get a moment of love, warmth, and happiness on a cold christmas eve.

Suppose we have m girls and n matches $(n \geq m)$.

In how many ways can we distribute the matches if

(a) Some girls may get 0 matches

(b) Every girl has to get at least one match.

Solution:

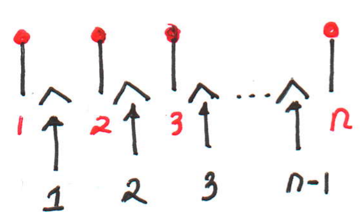
(a) We have already dealt with this situation!

$$\binom{n+m-1}{n}$$

(b) Here we are interested in the positive integer solutions to the equation

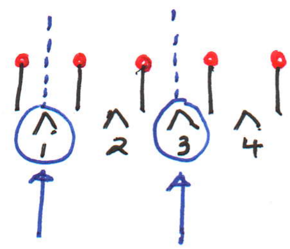
$$X_1 + X_2 + \dots + X_m = n$$
$$X_k \geq 1.$$

We may reason as follows:



There are $n-1$ spaces between the matches. If we choose $m-1$ spaces, we achieve an m-fold partition.

For example, suppose we have $m = 3$ girls and $n = 5$ matches. Then



means $X_1 = 1$, $X_2 = 2$, $X_3 = 2$.

In particular, with m girls and n matches we have $\binom{n-1}{m-1}$ possible partitions that don't leave any girl completely in the cold.

The General Approach

The number of positive integer solutions to the equation $X_1 + X_2 + \ldots + X_m = n$

$$X_k \geq 1$$

is $\binom{n-1}{m-1}$. What should be the number of solutions if we require $X_k \geq n_k$?

Observe that $X_k \geq n_k$ is equivalent to $Y_k = X_k - (n_k - 1) \geq 1$

and the equation

$$X_1 + X_2 + \ldots + X_m = n$$

is equivalent to

$$\left(X_1 - (n_1 - 1)\right) + \left(X_2 - (n_2 - 1)\right) + \ldots + \left(X_m - (n_m - 1)\right) =$$

$$= n - (n_1 - 1) - (n_2 - 1) - \ldots - (n_m - 1)$$

$$= n + m - n_1 - n_2 - \ldots - n_m$$

Setting $y_k = X_k - (n_k - 1)$ we obtain the equation

$$\underset{y_k \geq 1}{y_1 + y_2 + \ldots + y_m} = n + m - n_1 - n_2 - \ldots - n_m$$

which has exactly

$$\binom{n + m - n_1 - n_2 - \ldots - n_m - 1}{m - 1}$$

distinct solutions.

Ex. You must pay the mob at least \$10,000 this week and your bank account is empty. You then recall the prized collection of silver spoons in your family. You own 2 spoons, your grandma has 6, and your parents have 4. If each spoon is worth \$1,000 and you intend to pay your dues with the spoons

(a) In how many ways can you do this while stealing as little as possible from your family?

(b) In how many ways, if stealing from family members isn't your primary concern?

Solution:

(a) We want to find the number of solutions to the equation $x_1 + x_2 + x_3 = 2$

$x_1 \geq -6, \ x_2 \geq -4, \ x_3 \geq 10.$

This number is $\binom{2+3+6+4-10-1}{2} = \binom{4}{2} = 6$

(b) Let X_1, X_2, X_3, X_4 be the number of spoons that remain in the hands of grandma, parents, mob, and yourself respectively.

We want to count the number of integer solutions to the equation

$$X_1 + X_2 + X_3 + X_4 = 12$$

$$X_1 \geq 0,\ X_2 \geq 0,\ X_3 \geq 10,\ X_4 \geq 0.$$

Clearly this number is

$$\binom{12+4-0-0-10-0-1}{3}$$

or $\binom{5}{3} = 10$