Combinatorial Analysis Lecture 5

Multinomial Coefficients

Ex. 3 dragons have each kidnapped a princess. 10 brave knights are sent to rescue them: 3 to the first princess, 3 to the second princess, and 4 to the last (smallest) princess. How many "destinies" are possible if all the knights have been eaten?

Solution: Number the knights 1–10

Now what?!

Since the knights are destined to be lunch, place them on 3 trays. Tray 1 has 3 places and will go to dragon 1, tray 2 also has 3 places and will go to dragon 2. Tray 3 with 4 places will be delivered to dragon 3.

$$\underbrace{\circ\ \circ\ \circ}_{1}\quad \underbrace{\circ\ \circ\ \circ}_{2}\quad \underbrace{\circ\ \circ\ \circ\ \circ}_{3}$$

There are 10! linear orderings of the knights along the trays. Within each tray, knights can be shifted around without altering their destiny. There are consequently $3! \cdot 3! \cdot 4!$ permutations to each destiny. Thus the number of destinies

$$\binom{10}{3\ 3\ 4} = \frac{10!}{3!\,3!\,4!}$$

In general, if we have n knights (n distinct objects) that have to be assigned among r dragons (r distinct categories) with $n_1 \mapsto$ dr 1 $\quad n_2 \mapsto$ dr 2, ..., $n_r \mapsto$ dr r

where $n_1 + n_2 + \ldots + n_r = n$, the number of destiny assignments is a multinomial coefficient

$$\binom{n}{n_1\, n_2 \ldots n_r} = \frac{n!}{n_1!\, n_2! \ldots n_r!}$$

<u>Expanding $(x_1 + x_2 + \ldots + x_m)^n$</u>

$$(x_1 + x_2 + \ldots + x_m)^n = (x_1 + x_2 + \ldots + x_m)_1 \ldots (x_1 + x_2 + \ldots + x_m)_n$$

$$= \sum \underset{1}{\square}\,\underset{2}{\square} \ldots \underset{n}{\square}$$

where the sum is over all possible matings of n elements, one from each parenthesis.

<u>Q.</u> How many additions in the above sum?

<u>A.</u> Each box is an experiment with m outcomes. Hence there are m^n summands.

Now if $n_1 + n_2 + \ldots + n_m = n$, there will be summands of the form $x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$.

How many such summands are there?

We can think of the parentheses as knights and $x_1, \ldots, x_m$ as the dragons; wherever a parenthesis is assigned to x_k, x_k is the term that it contributes.

Exactly n_1 parentheses are delegated to x_1, n_2 - to x_2, ..., n_m - to x_m.

There are $\binom{n}{n_1\ n_2 \ldots n_m}$ ways to do that.

Thus $(x_1 + x_2 + \ldots + x_m)^n =$

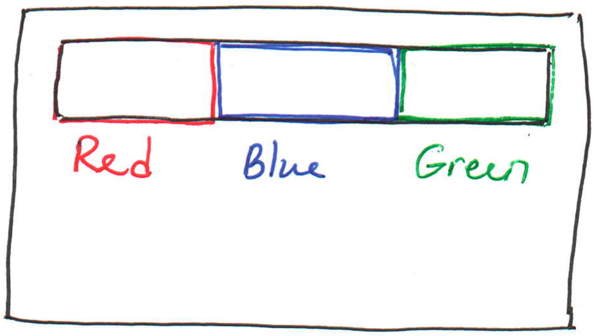
$$= \sum_{\substack{(n_1, n_2, \ldots, n_m): \\ n_1 + n_2 + \ldots + n_m = n}} \binom{n}{n_1\ n_2 \ldots n_m} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$$

Ex. (a) 9 pupils are divided into 3 teams, each consisting of 3 players. If one team will wear a red shirt, another – blue, and last – green, how many outcomes are possible?

(b) Only red shirts arrived. 3 players will be assigned red jersey and the remaining teams will not be wearing distinguishing colors. How many outcomes are possible now?

Solution: Number pupils 1–9.

(a) Kafka form

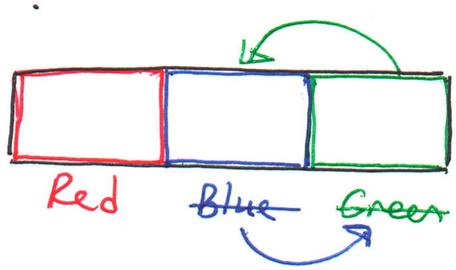


Protocol: Numbers within each color must be listed in order. For example if 253 → Red 9,1,4 → Blue 786 → Green, we input 235 149 678

Clearly there are exactly $\binom{9}{3\ 3\ 3} = \frac{9!}{(3!)^3}$

ways to fill out that form.

(b) Since we no longer have blue or green shirts, Blue and Green cannot be distinguished from one another:



For example, the code

235. 149. 678 $=$ 235. 149. 678 $\Longleftrightarrow$

235. 678. 149

and these codes mean exactly the same thing.

In other words, each distinguishable team assignment has exactly 2 documents out of $\binom{9}{3\ 3\ 3}$ documents that describe it. Hence the desired number is

$$\frac{1}{2}\binom{9}{3\ 3\ 3} = \frac{9!}{2\,(3!)^3}$$

Comprehension Check

Suppose that no shirts arrived. How many possibilities now?

Solution:

The 3 chunks of code can be permuted among themselves without changing the outcome that they describe. Hence there are exactly

$$\frac{1}{3!}\binom{9}{3\ 3\ 3} = \frac{9!}{(3!)^4}$$

different outcomes.