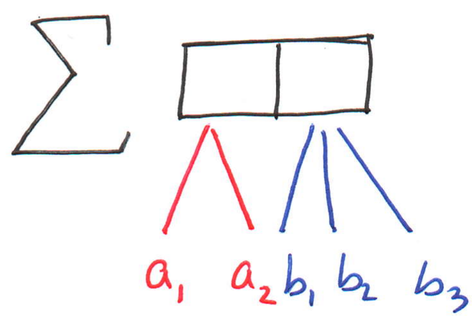
Combinatorial Analysis Lecture 4

The Binomial theorem

Observe that foiling results in a sum, where each summand is a unique product of representative terms from each parentheses.

For example

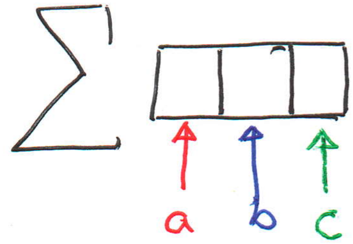
$$(a_1+a_2)(b_1+b_2+b_3) = a_1b_1 + a_1b_2 + a_1b_3$$
$$+\ a_2b_1 + a_2b_2 + a_2b_3$$
$$=$$



where each combination $a_i b_j$ occurs exactly once. (For a total of $2 \cdot 3$ summands).

Similarly

$$(a_1+a_2)(b_1+b_2+b_3)(c_1+c_2+c_3+c_4) =$$



How to expend $(x+y)^n$?

1) $(x+y)^1 = x+y$

2) $(x+y)^2 = (x_1 + y_1)(x_2 + y_2)$

$$= \sum \boxed{}\boxed{}$$

index 1 $(\text{''})_1$ index 2 $(\text{''})_2$

$$= \underbrace{\ \ 1\ \ }_{\substack{0\,x \\ \uparrow \\ y_1 y_2}} y^2 + \underbrace{\ \ 2\ \ }_{\substack{1\,x \\ \uparrow \\ x_1 y_2 + y_1 x_2}} xy + \underbrace{\ \ 1\ \ }_{\substack{2\,x \\ \uparrow \\ x_1 x_2}} x^2$$

$\vdots$

n) $(x+y)^n = (x_1+y_1)(x_2+y_2)\dots(x_n+y_n)$

$$= \sum \boxed{}\boxed{}\boxed{\ \cdots\ }\boxed{}$$

$(\text{''})_1 \quad (\text{''})_2 \quad \dots \quad (\text{''})_n$

Thus there are 2^n summands, which can be grouped according to the number of x:

$$(x+y)^n = \underline{\qquad}\, y^n + \underline{\qquad}\, x y^{n-1}$$
$$+ \dots + \underline{\qquad}\, x^k y^{n-k} + \dots + \underline{\qquad}\, x^n$$

The coefficient of $x^k y^{n-k}$ is just the number of summands in which exactly k of the n parentheses have contributed an x. This clearly corresponds to the number of subsets of size k (i.e. the specific k parentheses which contributed x) out of a set of size n (the n parentheses). Thus the coefficient of $x^k y^{n-k}$ is $\binom{n}{k}$.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1}$$
$$+ \dots + \binom{n}{n} x^n y^{n-n}.$$

Comprehension Check

Find the coefficient next to x^2y^2 in the expansion of $(x+y)^4$.

Solution:

$$(x+y)^4 = (x+y)_1(x+y)_2(x+y)_3(x+y)_4$$

x^2y^2 means that 2 of the 4 parentheses have contributed an x and the other 2 parentheses have contributed a y. The possibilities are

$$\underset{1}{x}\cdot\underset{2}{x}\cdot\underset{3}{y}\cdot\underset{4}{y}, \quad \underset{1}{x}\cdot\underset{2}{y}\cdot\underset{3}{x}\cdot\underset{4}{y}, \quad \underset{1}{x}\cdot\underset{2}{y}\cdot\underset{3}{y}\cdot\underset{4}{x}$$

$$\underset{1}{y}\cdot\underset{2}{x}\cdot\underset{3}{x}\cdot\underset{4}{y}, \quad \underset{1}{y}\cdot\underset{2}{y}\cdot\underset{3}{x}\cdot\underset{4}{x}, \quad \underset{1}{y}\cdot\underset{2}{x}\cdot\underset{3}{y}\cdot\underset{4}{x}$$

Rather than to list all the possibilities, simply note that it is enough to count the number of ways to select 2 out of 4 parenthesis to each contribute an x. $\binom{4}{2} = \frac{4!}{2! \cdot 2!} = 3! = 6$.

Comprehension Check

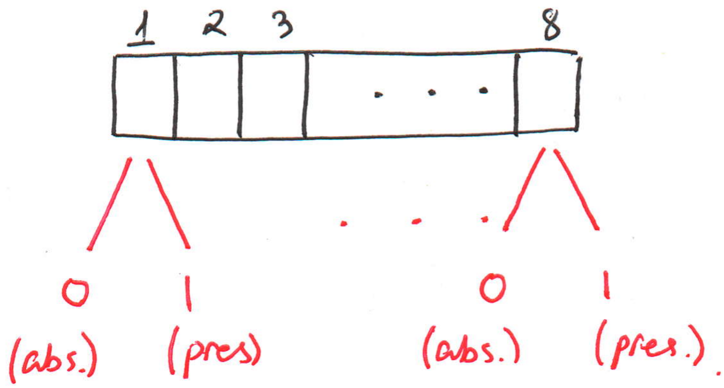
Calculate $\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \ldots + \binom{8}{8}$

Solution:

1) Observe that this is $\sum_{k=0}^{8} \binom{8}{k} 1^k \cdot 1^{8-k}$

$= (1+1)^8 = 2^8 = 256$

2) If I have 8 students, I can rank them 1-8 and take attendence



There are 2^8 attendence rosters.

We can group these rosters into (unequal) piles based on the number of students that attended. The pile designated "k students came" contains $\binom{8}{k}$ documents.

Summing over all the piles, we obtain

$$2^8 = \binom{8}{0} + \binom{8}{1} + \dots + \binom{8}{8}$$

In general, $2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$.

<u>Useful identity</u>

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Verifying this with algebra is tedious and unenlightening, but we can figure it out via combinatorial reasoning:

In a class of n individuals, Jimmy, designated with the number n is special. The remaining kids are each designated with the numbers $1 - n-1$. r of the children are selected to go on a field trip. The Kafka form on which the selected kids are recorded looks like this:

$$\boxed{1\;3\;4 \ldots r+1}$$

$\longrightarrow$ list r numbers in inc. order

Here, for example, children #$1, 3, 4, 5, \ldots, r+1$ were selected to go on the trip.

Clearly there are $\binom{n}{r}$ ways to fill in this form (once r out of n numbers are selected, there is only one way to list them in ascending order)

Now divide the forms into two piles:

<u>Pile 1:</u> Jimmy picked   <u>Pile 2:</u> Jimmy not.

These piles are generally not of the same size

Pile 1

$$\begin{array}{cccc} 1 & 2 & \ & r-1 & r \\ \hline * & * & \dots & * & n \end{array}$$

$\longrightarrow$ inc. order

There are $r-1$ numbers that must be selected from $n-1$ numbers $\Rightarrow$

$$\binom{n-1}{r-1}$$

possibilities.

Pile 2

$$\begin{array}{ccccc} 1 & 2 & \ & r-1 & r \\ \hline * & * & \dots & * & * \end{array}$$

$\longrightarrow$ inc. order

r numbers must be picked. n is off limits! $\Rightarrow$ Pick r from $n-1$ numbers:

$$\binom{n-1}{r}$$

possibilities.

Simply put

$$\binom{n}{r} = \underbrace{\binom{n-1}{r-1}}_{\text{jimmy in}} + \underbrace{\binom{n-1}{r}}_{\text{jimmy out}}.$$

Pascal's Triangle

$$\begin{array}{ccccccccccl}
 & & & & 1 & & & & & & \\
 & & & 1 & & 1 & & & & & \Longleftrightarrow (x+y)^1 \\
 & & 1 & & 2 & & 1 & & & & \Longleftrightarrow (x+y)^2 \\
 & 1 & & 3 & & 3 & & 1 & & & \Longleftrightarrow (x+y)^3 \\
\underline{1} & & \underline{4} & & \underline{6} & & \underline{4} & & \underline{1} & & \Longleftrightarrow (x+y)^4
\end{array}$$

Pascal's Triangle makes use of the identity

$\binom{n-1}{r-1} + \binom{n-1}{r}$ to calculate the next row

from the previous one.

For instance

$$\begin{array}{ccc}
1 & 2 & 1 \\
\| & \| & \| \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2}
\end{array}$$

tells us that $\binom{3}{1} = \binom{3-1}{1-1} + \binom{3-1}{1} = \binom{2}{0} + \binom{2}{1}$

$= 1 + 2 = 3.$

Thus, for instance $(x+y)^4 = \sum_{k=0}^{4} \binom{4}{k} x^{4-k} y^k$

$= 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

Comprehension Check

Use a combinatorial argument (tell a story) to simplify the formula

$$\binom{n}{r}\binom{m}{0} + \binom{n}{r-1}\binom{m}{1} + \dots + \binom{n}{0}\binom{m}{r}$$

$$= \sum_{k=0}^{r} \binom{n}{r-k}\binom{m}{k}$$

Solution:

Suppose a class has n girls and m boys. r pupils will be summoned to the director's office for bad behavior. $\binom{n}{r-k}\binom{m}{k}$ – number of ways to summon k boys.

Clearly the number of boys is between 0 and r

Thus $$\sum_{k=0}^{r} \binom{n}{r-k}\binom{m}{k} = \binom{n+m}{r}$$