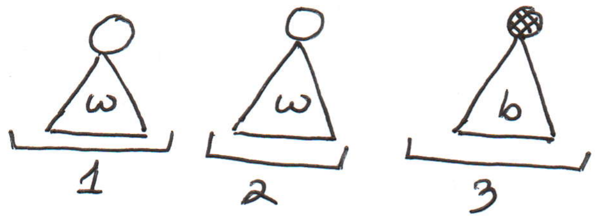
Combinatorial Analysis Lecture 2

How to deal with Overcounting

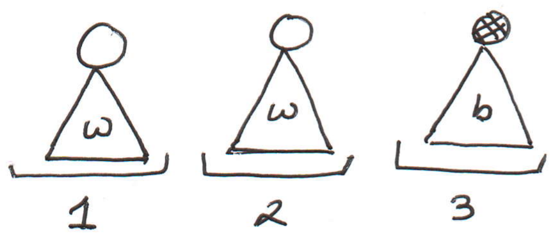
Ex.



Two white and one black pawn.
Any two pawns of the same color are indistinguishable
How many visually distinct linear patterns are possible?

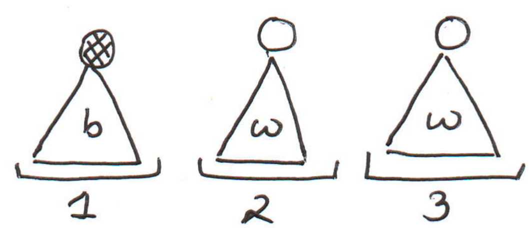
Solution: Clearly the linear pattern is completely determined by the placement of the black pawn.

1)



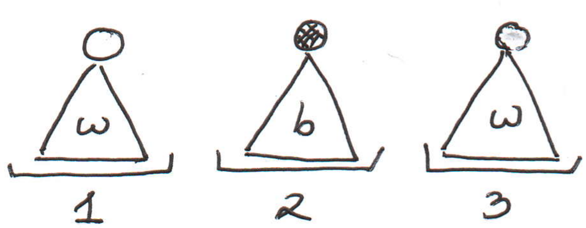
(w, w, b)

3)



(b, w, w)

2)

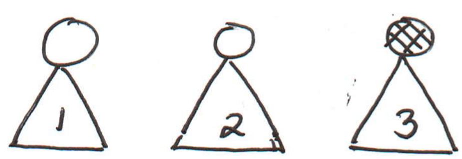


(w, b, w)

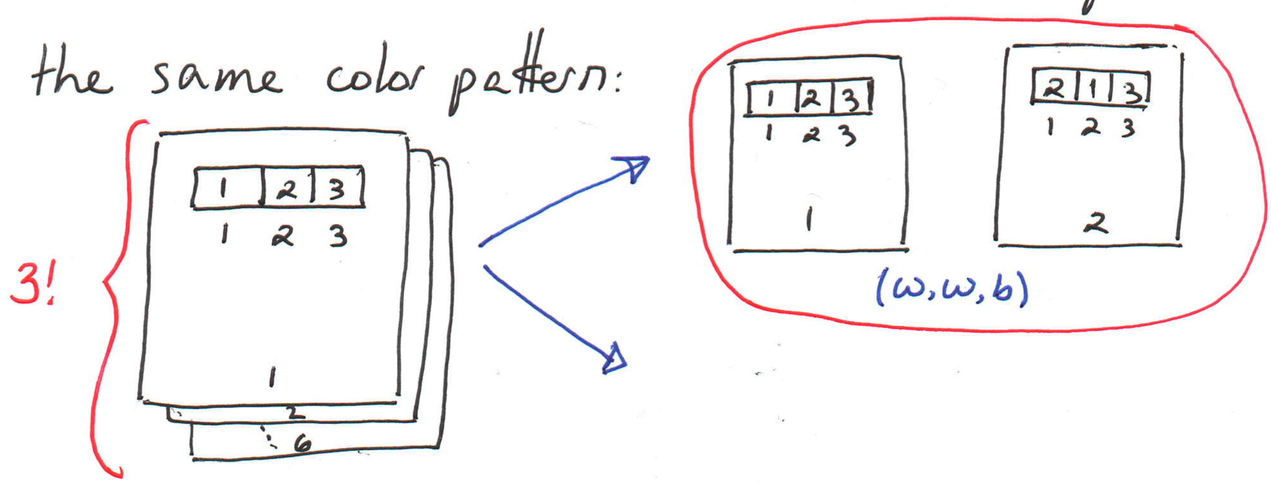
In this simple example, we were able to get by by means of direct counting. As our examples get more complex, we will rarely be as lucky.

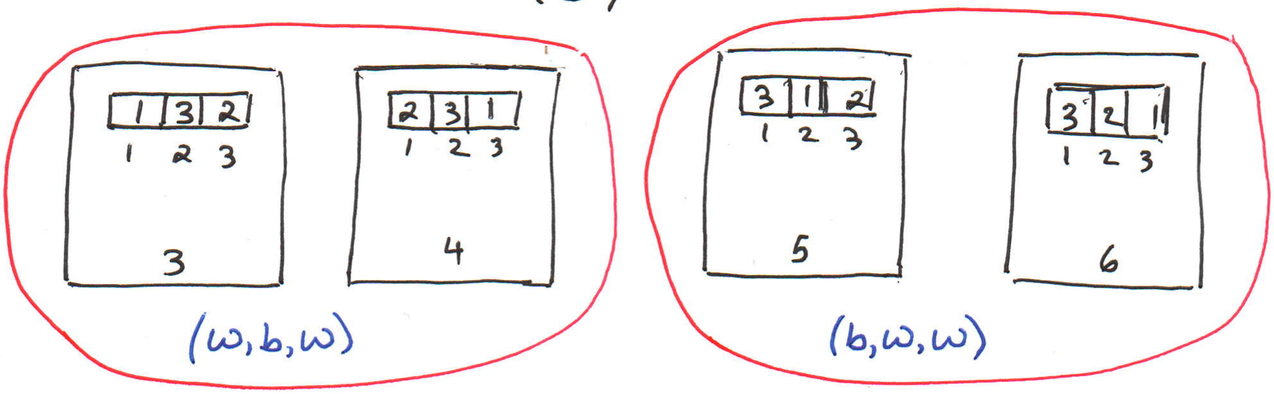
Let's try to apply the Kafka method.

I often have a great difficulty distinguishing between students. To help myself imagine that students (or people in general) are unique, I picture that each individual has swallowed an integer that is uniquely in his belly. Let's imagine the pawns did the same:



If we print out a Kafka protocol for each linear ordering of the pawns as individuals, we get 3! documents, some of which correspond to the same color pattern:





Observe that a bundle of documents, rather than the documents themselves now encode a distinguishable outcome. Notice also that each bundle has the same number of documents. Hence

$k \equiv$ number of distinguishable linear patterns

and

$j \equiv$ number of documents per bundle $= 2$

satisfy

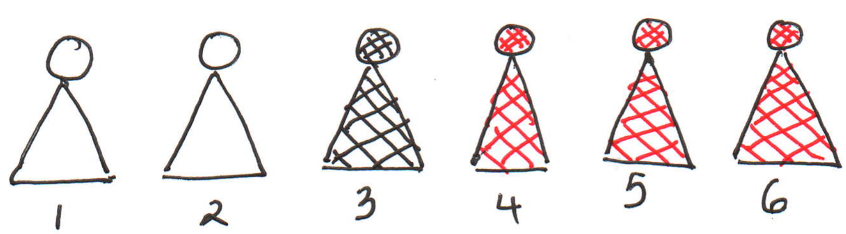
$$k \cdot j = 3!$$

Hence $k = \frac{3!}{j} = \frac{3!}{2} = \frac{6}{2} = 3.$

Again: (# piles) $\cdot$ (# per pile) $=$ (# documents)

$$\boxed{\#\text{distinguishable outcomes} = \#\text{piles} = \frac{\#\text{doc.}}{\#\text{per pile}}}$$

Ex. 6 pawns with any two pawns of the same color being indistinguishable are to be arranged in a row. How many distinguishable patterns are there if the pawns are as follows



Solution:

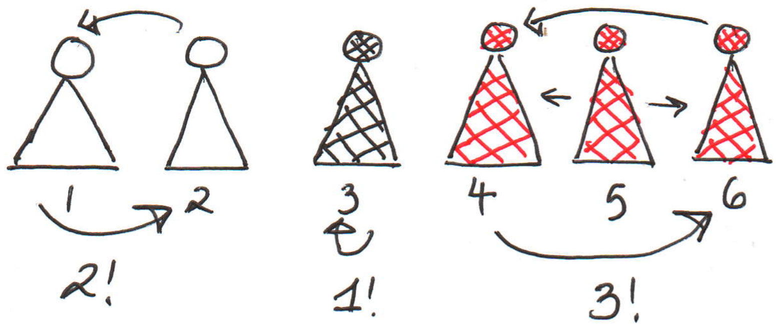
Kafka form:

1	2	3	4	5	6

There are a total of $6!$ ways to fill out this form. For instance

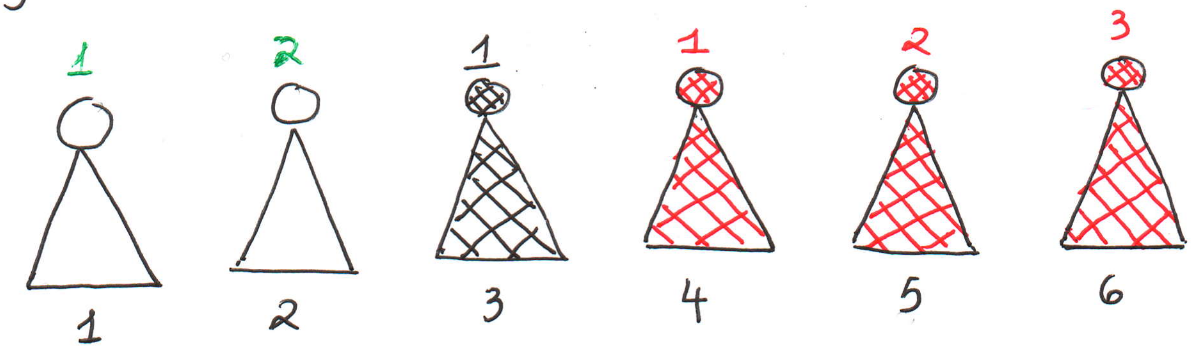
1	2	3	4	5	6
1	2	3	4	5	6

corresponds to the pattern (w, w, b, r, r, r). How many more documents identify this pattern? Well, we can rearrange the whites among themselves, the blacks among themselves, and the reds among themselves without altering the pattern.



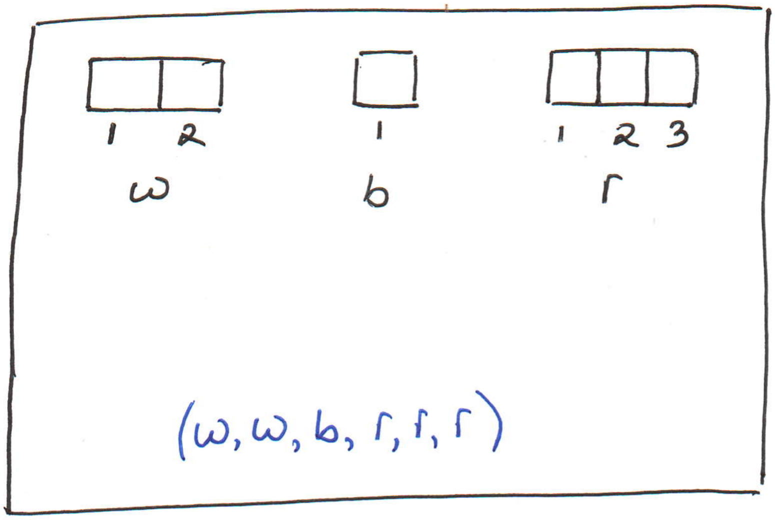
The total number of rearrangements that don't alter the pattern is $2! \cdot 1! \cdot 3!$

Don't believe me? Good! In this day and age not only your pawn number, but its color ranking plays a vital role. So:



Imagine that each pawn carries two numbers that indicate its overall ranking and its color ranking respectfully.

For each color pattern devise a form that looks as follows:



This form records the order in which the color indices are arranged among themselves.
For instance, the code 2 1 1 3 1 2 on this form indicates that the pattern (w, w, b, r, r, r) was produced by the pawns arranged linearly as 2 1 3 6 4 5. By the basic principle of counting there are $2! \cdot 1! \cdot 3!$ ways to fill out this form. Thus, exactly $2! \cdot 1! \cdot 3!$ of the $6!$ permutations of pawns correspond to the pattern (w, w, b, r, r, r).

For any other pattern we can use the same type of form. For example, writing the code 2 1 1 3 1 2 on the document labled as (w, r, r, w, b, r) indicates that the pawns are in linear order 2 6 4 1 3 5

Thus if we group the documents that correspond to linear arrangements of pawns that are visually alike, we get a pile that is $2!\cdot 1!\cdot 3!$ documents thick. Since this is true of every pile

We have

$$k = \#\text{piles} = \frac{\#\text{documents}}{\#\text{num per pile}} = \frac{6!}{2!\,1!\,3!}$$

<u>Ex.</u> How many rearrangements of the word NECESSARY are there

<u>Solution:</u>

N - #1

E - #2

C - #1

S - #2

A - #1

R - #1

Y - #1

$$\frac{9!}{1!\,2!\,1!\,2!\,1!\,1!\,1!} = \frac{9!}{2!\,2!}$$

Ex. 10 competitors. 4 - Russian, 3 - US, 2 - British 1 - Brasilian. How many outcomes are possible if the result of this competition reports the ranking of each player by nationality?

Solution:

An outcome might look like this:

$$\begin{pmatrix} R & U & U & R, & B, & U, & R, & R, & Br, & Br \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

Clearly there are $\dfrac{10!}{4!\,3!\,2!\,1!}$

The general principle is as follows. If n objects are to be arrayed linearly, n_1 of which are indistinguishable among themselves, n_2 of which are indistinguishable among themselves etc... $(n_1 + n_2 + \ldots + n_k = n)$

The total number of linear orderings is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$