# Combinatorial Analysis Lecture 1

## Basic Principle of Counting

Ex. 1) Coin flipped. Two possible outcomes

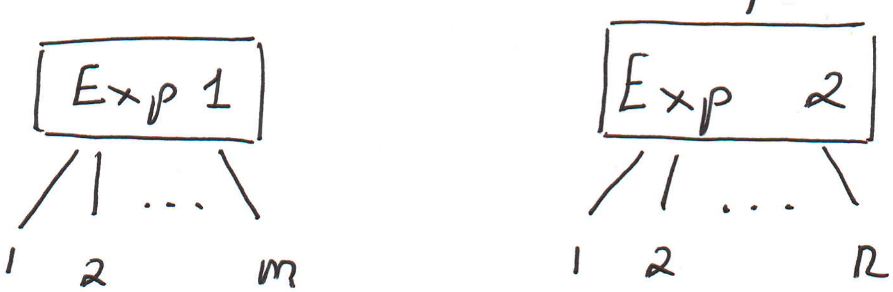
H   T

2) Coin flipped and die tossed. Possible outcomes are

(H, 1) (H, 2) (H, 3) (H, 4) (H, 5) (H, 6)

(T, 1) (T, 2) (T, 3) (T, 4) (T, 5) (T, 6)

In general, if 2 experiments are performed



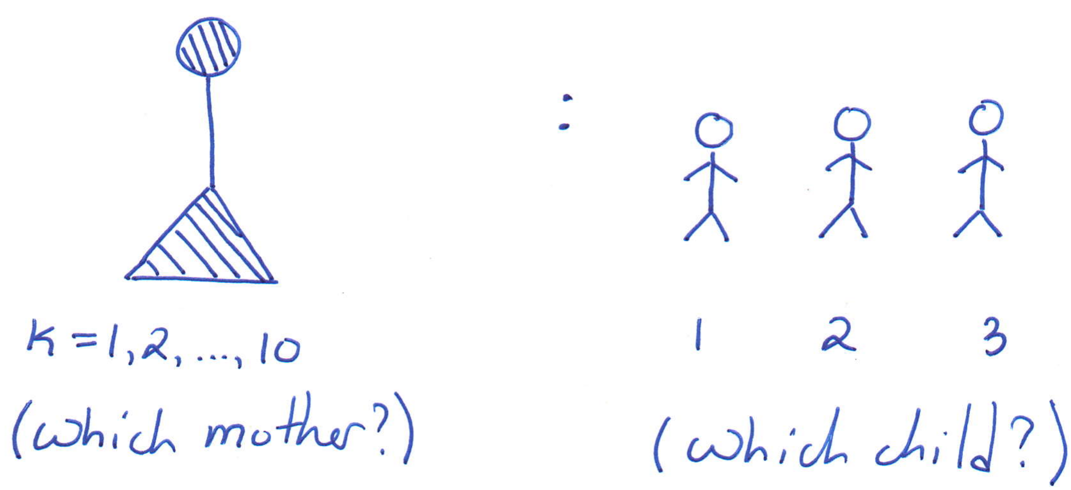

the total set of possible outcomes can be listed as

$$\begin{array}{cccc}
(1,1) & (1,2) & \cdots & (1,n) \\
(2,1) & (2,2) & \cdots & (2,n) \\
\vdots & \vdots & & \vdots \\
(m,1) & (m,2) & \cdots & (m,n)
\end{array}$$

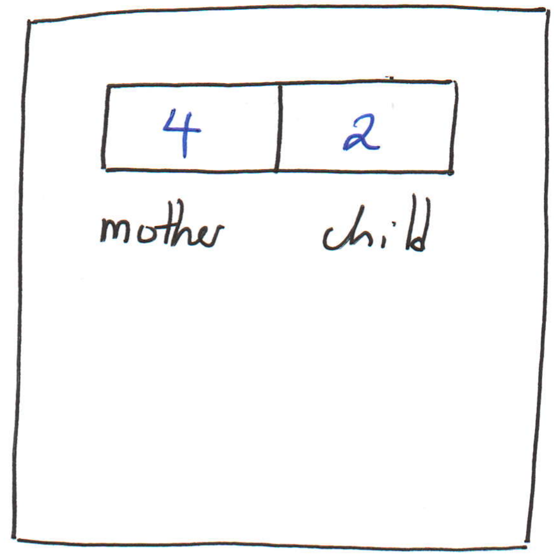
Thus, we have a matrix with $m$ rows and $n$ columns. The total number of entries is thus $m \cdot n$.

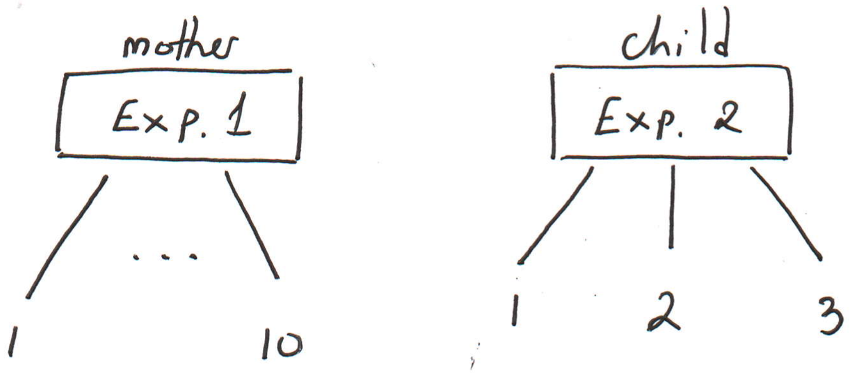
<u>Ex.</u> There are 10 women. Each woman has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

<u>Solution:</u> Although this may seem a very simple problem, in order to avoid making errors, you should try to think about every combinatorial problem through the eyes of a bureaucrat, devising a protocol for every given situation. I call them Kafka documents.



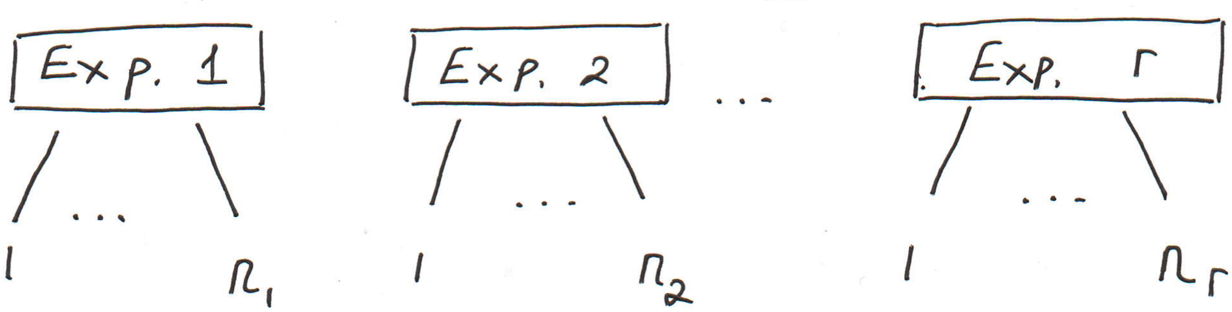
The document



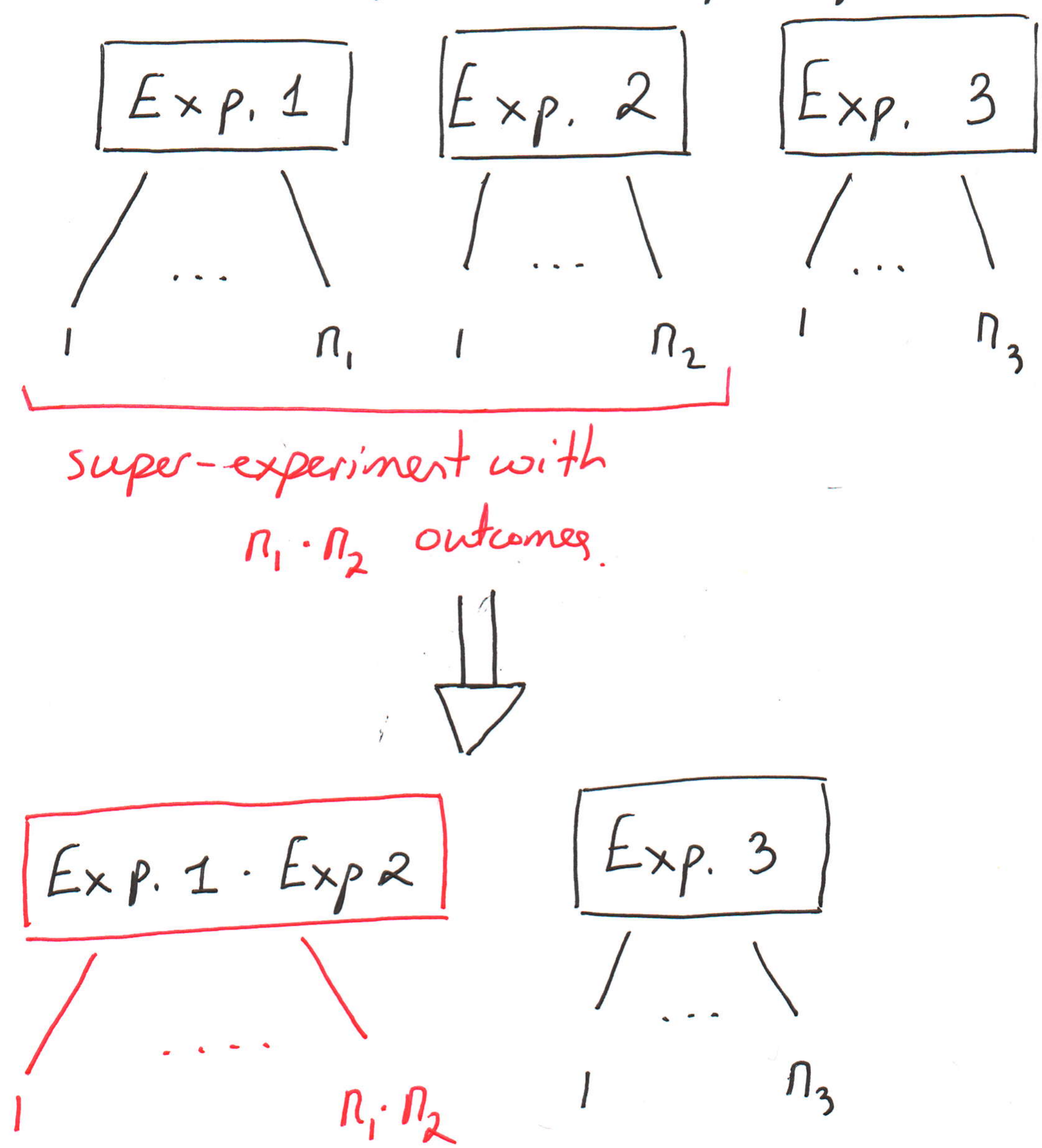




By the basic principle of counting, we have $10 \cdot 3 = 30$ ways of filling the document. Hence 30 possibilities of picking mother and child.

Generalized Principle of counting



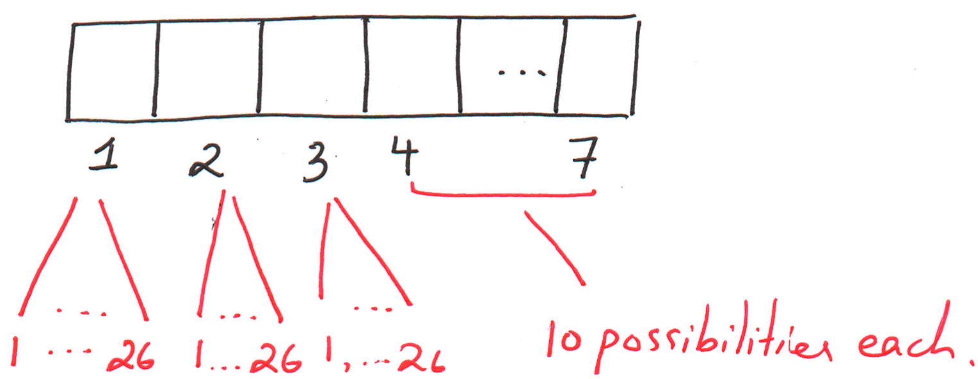
Using induction, observe that experiments 1 and 2 may be thought of as one super-experiment with $n_1 \cdot n_2$ outcomes (by the basic principle of counting).




Applying the basic principle of counting again, we obtain the super-super-experiment Exp1 · Exp2 · Exp3, consisting of $n_1 \cdot n_2 \cdot n_3$ outcomes. Continuing in this fashion, we see that the total set of possibilities has $n_1 \cdot n_2 \cdot \ldots \cdot n_r$ outcomes.

<u>Ex.</u> How many different 7-plate license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

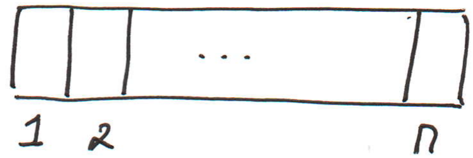
<u>Solution:</u> The kafka protocol will identify exactly how the generalized principle of counting is being used!




$26^3 \cdot 10^4$ possibilities.

<u>Ex.</u> How many functions defined on $n$ points are possible if each functional value is either 0 or 1?

Solution:

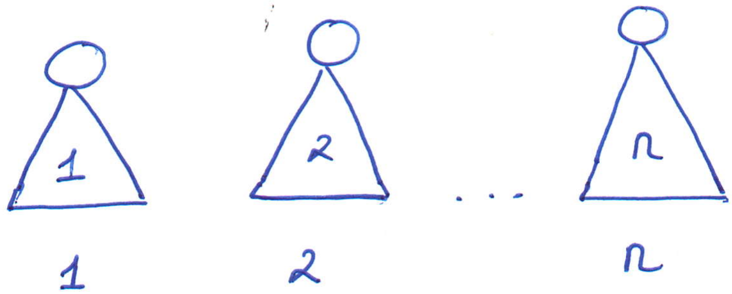



Thus there are $n$ experiments, each having 2 possible outcomes.
By the generalized principle of counting, we have

$\underset{1}{2}\cdot\underset{2}{2}\cdot\ldots\cdot\underset{n}{2} = 2^n$ outcomes.

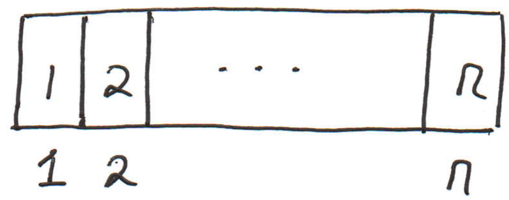
## Permutations

How many ways are there to arrange $n$ students in a raw?

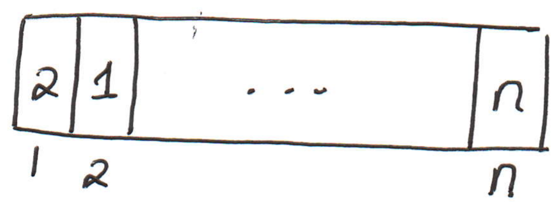



Solution: This philosophy may prove useful:
Two objects occupying distinct spaces at the same time are distinct (even when you cannot distinguish between them!). Here it means that the students are all distinct.

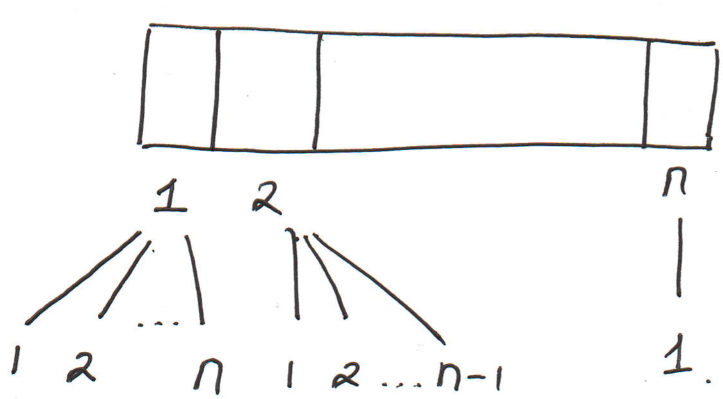
I myself often have trouble telling my students apart, so I imagine each swallowed an integer.




Means student #1 in first place on linear grid, student #2 in second place ... student #$n$ in $n^{th}$ place. On the other hand, if I see the arrangement $2, 1, 3, 4, 5, 6, \dots, n$, I can record it in my protocol as




Clearly there is a 1-1 correspondence between codes in my protocol and raw arrangements.



Thus there are $n \cdot (n-1) \cdot \ldots \cdot 1 = n!$ possibilities.

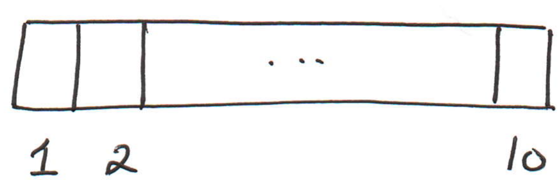
<u>Ex.</u> Class has 6 men and 4 women. Students are ranked according to their performance, from best to worst.

(a) How many different rankings are possible?

(b) How many rankings if men are ranked among themselves and women are ranked among themselves?

<u>Solution:</u>

(a) Kafka document could be



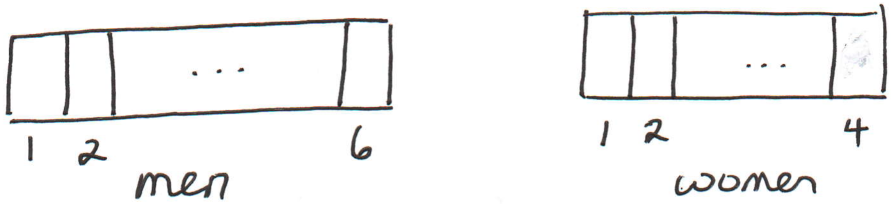

where everyone is assigned one integer 1-10 and placed on the grid according to rank.

3 2 1 4 5 6 7 8 9 10, for istance, corresponds to student #3 being best, #2 second best etc.

Clearly there are $10!$ possibilities.

(b) Label the men 1-6 and women 1-4

The Kafka protocal can look like this:




By applying the generalized principle of counting several times, we see that there are $6! \cdot 4!$ possibilities to fill out this document.

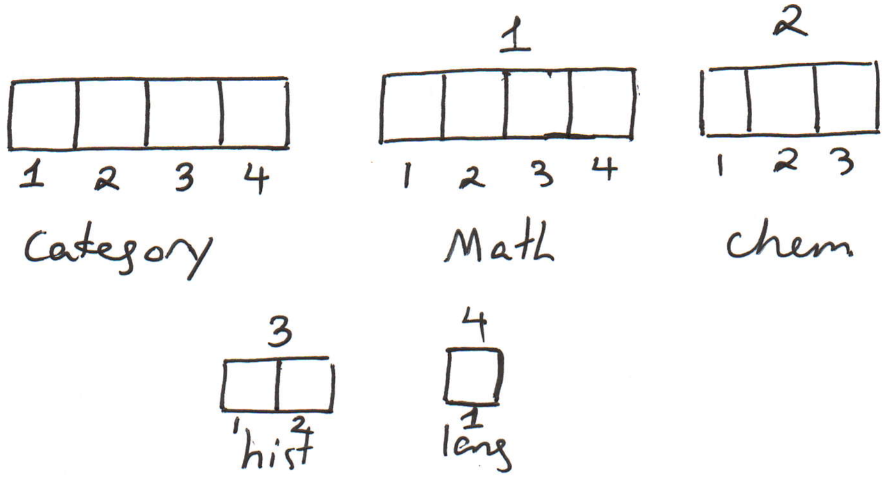
Note: $6! \cdot 4! = 4! \cdot 6!$ But the latter calculation indicates a different version of the Kafka document - One in which women were listed first.

Ex. You wish to arrange 10 books on a shelf. 4-math books, 3-chemistry, 2-history, and 1-language book.
Books on the same subject are to be kept together. How many arrangements are possible?

Solution: Assign codes to different categories

1 = math, 2 = chem, 3 = hist. 4 = lang.

Document




within each category, we assign volume 1, 2, ...,

Thus a code like:

2 4 1 3    1 2 3 4    1 2 3    1 2    1

means that the volumes are arranged in order within each subject and as I walk left to right along the shelf, I see chem → lang → math → hist.

Clearly each code identifies a unique arrangement and each arrangement corresponds to a unique code.

Applying the generalized principle of counting, we get

$4! \cdot 4! \cdot 3! \cdot 2! \cdot 1!$