Solutions to Complex Variables

Exam 3

1. Let $f(z) = a_0 + a_1 z + \dots + a_n z^n$ be a polynomial with coefficients in $\mathbb{C}$. Show that for all but finitely many $w \in \mathbb{C}$, $f(z) - w$ has n distinct roots in $\mathbb{C}$.

Solution: Suppose z_w is a root of order $k \geq 2$ of $f(z) - w = 0$. We make the following observations

(i) If z_w is also a root of $f(z) - \xi = 0$ we have

$$\begin{cases} f(z_w) - w = 0 \\ f(z_w) - \xi = 0 \end{cases}$$

$\Rightarrow w = \xi$. Hence $f^{-1}\{w\} \cap f^{-1}(\xi) = \emptyset$ unless $w = \xi$ (which is also obvious as f is a function).

(ii) We can write $f(z) = (z - z_w)^k h_w(z)$ where $h_w(z)$ is a polynomial of degree $n-k$ and $h_w(z_w) \neq 0$.

Notice that $f'(z_w) = \frac{d}{dz}\left(f(z) - w\right)\Big|_{z = z_w} =$

$$= k(z - z_w)^{k-1} h_w(z) + (z - z_w)^k h_w'(z)\Big|_{z = z_w} = 0$$

Hence the zeros of $f(z) - w$ of multiplicity $k \geq 2$ are also the zeros of $f'(z)$. Since there could be at most $n-1$ distinct roots of $f'(z)$, There are at most $n-1$ w

for which $f^{-1}\{w\}$ contains fewer than n elements.

2. Let $f: \mathbb{C} \mapsto \mathbb{C}$ be entire and suppose $f(S') = S'$, where S' is the unit circle. Show that $f(z) = \alpha z^d$, for some $d \in \mathbb{N}$ and $|\alpha| = 1$.

Solution: Let $\omega_1, \dots, \omega_d$ be the zeros of f in $\mathbb{D} =$
$= \{z \in \mathbb{C} : |z| < 1\}$. Let $g_{\omega_k}(z) = \dfrac{z - \omega_k}{1 - \overline{\omega_k} z}$.

Since $f(S') \subseteq S'$ we have $|f(z)| = 1$ for all $z \in S'$.

Define $K(z) = \dfrac{f(z)}{g_{\omega_1}(z) \cdots g_{\omega_d}(z)}$. Then $|K(z)| = 1$ for

all $z \in S'$ and $|K(z)| > 0$ for all $z \in \overline{\mathbb{D}}$. By maximum modulus principle, $|K(z)| \leq 1$ on $\mathbb{D}$. In particular, $|K(z)| = 1$ on S'. If $\min_{z \in \overline{\mathbb{D}}} |K(z)| = 1$, Then $|K(z)| = 1$ on $\overline{\mathbb{D}}$, otherwise the minimum is taken at the point $z_0 \in \mathbb{D}$. Since $K(z) \neq 0$ on $\mathbb{D}$, we can then employ the minimum modulus principle to conclude that

$K(z) = \alpha$ for some $\alpha \in \mathbb{C}$.

It follows that for all $z \in \mathbb{D}$, $f(z) = \alpha g_{\omega_1}(z) \cdots g_{\omega_d}(z)$.

But then f is the analytic continuation of $\alpha g_{\omega_1}(z) \cdots g_{\omega_d}(z)$ to $\mathbb{C}$. This implies $\dfrac{z - \omega_k}{1 - \overline{\omega} z} = \infty$ iff $z = \infty \Rightarrow \omega_k = 0$

Hence $f(z) = \alpha z^d$ and since $\left|\frac{f(z)}{z^d}\right| = 1 = |\alpha|$, we know $\alpha = e^{i\theta}$.

3. Show that for any holomorphic function $f: \mathbb{D} \mapsto \mathbb{D}$

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}\, f(z_2)} \right| \leqslant \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|$$

for all $z_1, z_2 \in \mathbb{D}$. Study the case of equality.

Solution: Fix $z_1 \in \mathbb{D}$ and define $g: \mathbb{D} \mapsto \mathbb{D}$ by

$g = \varphi_{f(z_1)} \circ f \circ \varphi_{z_1}$, where $\varphi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ for $|\alpha| < 1$.

Notice that $g(0) = 0$. Hence, by Schwarz Lemma,

$|g(z)| \leqslant |z|$ for all $z \in \mathbb{D}$. In particular,

$$\left| \frac{f(z_1) - f(z)}{1 - \overline{f(z_1)}\, f(z)} \right| = \left| \varphi_{f(z_1)}(f(z)) \right| = \left| g(\varphi_{z_1}(z)) \right| \leqslant$$

$$\leqslant |\varphi_{z_1}(z)| = \left| \frac{z_1 - z}{1 - \bar{z}_1 z} \right|$$

By Schwarz lemma, if there exists $z_2 \in \mathbb{D}$ such that

$|g(z_2)| = |z_2|$, $|g(z)| = |z|$ identically for all z.

This shows that if $\left| \dfrac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}\, f(z_2)} \right| = \left| \dfrac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|$

then equality holds for all pairs (w_1, w_2).

4. Show that for any holomorphic function $f : \mathbb{D} \mapsto \mathbb{D}$

$$\frac{|f'(z)|}{1-|f(z)|^2} \leqslant \frac{1}{1-|z|^2}$$

for all z in the unit disc $\mathbb{D}$.

Solution: let $\varphi_\alpha(z) = \frac{\alpha - z}{1-\bar{\alpha} z}$ (for $|\alpha|<1$). Then $\varphi_\alpha'(z) = \frac{|\alpha|^2-1}{(1-\bar{\alpha}z)^2}$ and $|\varphi_\alpha'(z)| = \frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2}$.

Now if $z_1 \in \mathbb{D}$ consider the map $g : \mathbb{D} \mapsto \mathbb{D}$ given by $g = \varphi_{f(z_1)} \circ f \circ \varphi_{z_1}$. Clearly $g(0)=0$. By Schwarz lemma, $|g'(0)| \leqslant 1$. Now $|g'(0)| = \frac{1-|f(z_1)|^2}{(1-\overline{f(z_1)}f(z_1))^2} \cdot |f'(z_1)| \cdot (1-|z_1|^2)$

$= \frac{1-|z_1|^2}{1-|f(z_1)|^2} |f'(z_1)| \leqslant 1$. Hence

$$\frac{|f'(z_1)|}{1-|f(z_1)|^2} \leqslant \frac{1}{1-|z_1|^2}$$

Since this calculation is true for all $z_1 \in \mathbb{D}$ we conclude

$$\frac{|f'(z)|}{1-|f(z)|^2} \leqslant \frac{1}{1-|z|^2} \quad \text{on } \mathbb{D}.$$

Equality holds everywhere as soon as it holds for a single point z_1. Otherwise the inequality is strict.

5. If $f(z)$ is holomorphic and $\operatorname{Im} f(z) \geq 0$ for $\operatorname{Im} z > 0$, show that

$$\frac{|f(z)-f(z_0)|}{|f(z)-\overline{f(z_0)}|} \leq \frac{|z-z_0|}{|z-\overline{z_0}|}$$

and

$$\frac{|f'(z)|}{\operatorname{Im} f(z)} \leq \frac{1}{\operatorname{Im} z}.$$

Solution: Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ denote the upper half plane. Define $\varphi_\alpha : \mathbb{H} \mapsto \mathbb{D}$ (for $\alpha \in \mathbb{H}$) by

$\varphi_\alpha(z) = \dfrac{z-\alpha}{z-\overline{\alpha}}$. Notice that $\varphi_\alpha(\alpha) = 0$ and that

$\varphi_\alpha'(z) = \dfrac{2i \operatorname{Im} \alpha}{(z-\overline{\alpha})^2}$. Define $g : \mathbb{D} \mapsto \mathbb{D}$ by

$g = \varphi_{f(z_0)} \circ f \circ \varphi_{z_0}^{-1}$. Then $g(0) = 0$. By Schwarz lemma, we know that $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. In particular

$$\frac{|f(z)-f(z_0)|}{|f(z)-\overline{f(z_0)}|} = |g(\varphi_{z_0}(z))| \leq |\varphi_{z_0}(z)| = \frac{|z-z_0|}{|z-\overline{z_0}|}$$

Also, $1 \geq |g'(0)| = \dfrac{2 \operatorname{Im} f(z_0)}{|f(z_0)-\overline{f(z_0)}|^2} \cdot |f'(z_0)| \cdot \dfrac{|z_0-\overline{z_0}|^2}{2 \operatorname{Im} z_0}$

which yields $1 \geq \dfrac{|f'(z_0)|}{2 \operatorname{Im} f(z_0)} \cdot 2 \operatorname{Im} z_0$. Hence

$\dfrac{1}{\operatorname{Im} z_0} \geq \dfrac{|f'(z_0)|}{\operatorname{Im} f(z_0)}$. Since this inequality is true for all $z_0 \in \mathbb{H}$. The general statement $\dfrac{1}{\operatorname{Im} z} \geq \dfrac{|f'(z)|}{\operatorname{Im} f(z)}$ holds.

6. Suppose f is a holomorphic automorphism of $\mathbb{D}$ such that f has two fixed points. Show that f must be the identity.

Solution: Let $\alpha, \beta \in \mathbb{D}$ be the two fixed points of f. Let $\varphi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$. Then $g = \varphi_\alpha \circ f \circ \varphi_\alpha$ is an automorphism of $\mathbb{D}$ that fixes $0 = \varphi_\alpha(\alpha)$ and $\varphi_\alpha(\beta) \neq 0$.

Since $g(0) = 0$, Schwarz lemma tells us that

$|g(z)| \leq |z|$. Furthermore, $g(\varphi_\alpha(\beta)) = \varphi_\alpha(\beta)$ so in fact

$|g(z)| = |z|$ and therefore $g(z) = e^{i\theta} z$. Again, using the fixed point $\varphi_\alpha(\beta)$, we see that $\varphi_\alpha(z) = e^{i\theta}\varphi_\alpha(z)$.

Consequently, $e^{i\theta} = 1$. Thus $g(z) = z$ and

$f(z) = \varphi_\alpha \circ g \circ \varphi_\alpha(z) = \varphi_\alpha^{\circ 2}(z) = z.$

7. Let $\mathbb{P}$ denote the right half-plane $\{z : Re(z) > 0\}$. If $f : \mathbb{P} \mapsto \mathbb{P}$ is holomorphic and $f(1) = 1$ show that

(i) $|f'(1)| \leq 1$ and (ii) $\left|\frac{f(z)-1}{f(z)+1}\right| \leq \left|\frac{z-1}{z+1}\right|$

Solution: Let $\sigma : \mathbb{P} \mapsto \mathbb{D}$ be given by $\sigma(z) = \frac{z-1}{z+1}$.

Then $\sigma(1) = 0$ and $\sigma'(z) = \frac{2}{(z+1)^2}$. Define $g : \mathbb{D} \mapsto \mathbb{D}$ by $g = \sigma \circ f \circ \sigma^{-1}$. Notice that $g(0) = 0$

(i) By Schwarz lemma and chain rule $|g'(0)| \leq 1$ and $g'(0) = \sigma'(1) \cdot f'(1) \cdot \frac{1}{\sigma'(1)} = f'(1)$. Thus $1 \geq |f'(1)|$.

(ii) $|g(z)| \leq |z|$ so $|\sigma(f(z))| \leq |\sigma(z)|$, which gives

$$\left|\frac{f(z)-1}{f(z)+1}\right| \leq \left|\frac{z-1}{z+1}\right|.$$

8. Does there exist a holomorphic function $f: \mathbb{D} \mapsto \mathbb{D}$ such that $f(1/2) = 3/4$ and $f'(1/2) = 2/3$?

Solution: Let $\alpha \in \mathbb{D}$ and $\varphi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$. Observe that $\varphi_\alpha'(\alpha) = \dfrac{|\alpha|^2 - 1}{(1-|\alpha|^2)^2} = \dfrac{-1}{1-|\alpha|^2}$.

Define $g: \mathbb{D} \mapsto \mathbb{D}$ by $g(z) = \left(\varphi_{f(\alpha)} \circ f \circ \varphi_\alpha\right)(z)$. Then $g(0) = 0$ and, by Schwarz lemma,

$$1 \geq |g'(0)| = \left|\varphi'_{f(\alpha)}(f(\alpha))\right| |f'(\alpha)| |\varphi_\alpha'(0)| = \frac{1-|\alpha|^2}{1-|f(\alpha)|^2} |f'(\alpha)|$$

Hence, since the above calculation is valid for all $\alpha \in \mathbb{D}$, we get

$$\frac{1-|f(z)|^2}{1-|z|^2} \geq |f'(z)|.$$

In the present case

$$\frac{1-|f(1/2)|^2}{1-|1/2|^2} = \frac{1-9/16}{1-1/4} = \frac{16-9}{16-4} = \frac{7}{12} < \frac{8}{12} = \frac{2}{3}$$

$|f'(1/2)|$ is too big to satisfy the inequality. Therefore no such f exists.

9. Is there a holomorphic function $f: \mathbb{D} \to \mathbb{D}$ such that $f(0) = \frac{1}{2}$ and $f'(0) = \frac{3}{4}$? If so, find such an f. Is it unique?

Solution: Again, we can begin by checking that the inequality

$$\frac{1-|f(z)|^2}{1-|z|^2} \geq |f'(z)|$$

Since $\dfrac{1-\frac{1}{4}}{1-0^2} = \frac{3}{4} = |f'(0)|$, the function f must be an automorphism of $\mathbb{D}$. Hence $f(z) = c\dfrac{\alpha - z}{1-\bar{\alpha}z}$ for some $\alpha \in \mathbb{D}$ and $|c| = 1$.

Now $f(0) = c\alpha = \frac{1}{2}$ and $f'(0) = c(|\alpha|^2 - 1) = \frac{3}{4}$.

If we assume $\alpha \in \mathbb{R}$, we obtain the equations

$$2 = \frac{1}{c\alpha}$$

$$-\frac{3}{4} = c(1-\alpha^2)$$

Cancelling out c, we get $\dfrac{1-\alpha^2}{\alpha} = -\dfrac{3}{2}$ or $2\alpha^2 - 3\alpha - 2 = 0$

Thus $(2\alpha+1)(\alpha-2) = 0 \Rightarrow \alpha = -\frac{1}{2}$, $c = -1$.

Hence $f(z) = \dfrac{\frac{1}{2}+z}{1+\frac{1}{2}z} = \dfrac{1+2z}{2+z}$

This $f(z)$ is unique, because $\dfrac{c(|\alpha|^2-1)}{c\alpha} = \frac{3}{4}\cdot 2 = \frac{3}{2}$ implies $\alpha \in \mathbb{R}$ and, consequently α is a root of $2\alpha^2 - 3\alpha - 2$.

10. Suppose $f: \mathbb{D} \mapsto \mathbb{D}$ is holomorphic. Show that

$$\frac{|f(0)| - |z|}{1 - |f(0)||z|} \leqslant |f(z)| \leqslant \frac{|f(0)| + |z|}{1 + |f(0)||z|}$$

Solution: define $g: \mathbb{D} \mapsto \mathbb{D}$ by $g = \varphi_{f(0)} \circ f$, where $\varphi_{f(0)}(z) = \frac{f(0) - z}{1 - \overline{f(0)}z}$. Since $g(0) = 0$, Schwarz lemma implies that

$$|g(z)| = \left| \frac{f(0) - f(z)}{1 - \overline{f(0)}\, f(z)} \right| \leqslant |z|.$$

Hence $|f(0)| - |f(z)| \leqslant |f(0) - f(z)| \leqslant |z| \left(\left| 1 - \overline{f(0)} f(z) \right| \right) \leqslant$
$\leqslant |z| \left(1 + |f(0)||f(z)| \right)$.

Thus $|f(0)| - |z| \leqslant \left(1 + |f(0)||z| \right) |f(z)|$, implying

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leqslant |f(z)|.$$

On the other hand,

$$\left| \frac{1 - \overline{f(0)}\, f(z)}{f(0) - f(z)} \right| \geqslant \frac{1}{|z|}$$

implies

$$1 + |f(0)||f(z)| \geq \frac{1}{|z|}\Big(|f(z)| - |f(0)|\Big) \quad \text{Hence}$$

$$|z| + |f(0)| \geq |f(z)| - |f(0)||z||f(z)| = \Big(1 - |f(0)||z|\Big)|f(z)|$$

$$\frac{|f(0)| + |z|}{1 - |f(0)||z|} \geq |f(z)|$$

11. Suppose $f : \mathbb{D} \mapsto \mathbb{D}$ is holomorphic such that $|f(z^2)| \geq |f(z)|$ for all $z \in \mathbb{D}$. Show that f is constant.

Solution: Since $\varphi_{f(0)} \circ f(0) = \varphi_{f(0)} \circ f(0^2) = 0$, we may assume $f(z) = a_1 z + a_2 z^2 + \dots +$ for all $z \in \mathbb{D}$. We show that the assumption $|f(z)| \leq |f(z^2)|$ implies that each $a_k = 0$.
If not, let a_k be the first coefficient s.t. $a_k \neq 0$.
Then $f(z) = a_k z^k + a_{k+1} z^{k+1} + \dots +$ and

$$f(z^2) = a_k z^{2k} + a_{k+1} z^{2k+2} + \dots$$

By an argument that replicates the proof of the Schwarz Lemma, we have

$$1 \geq \frac{|f(z)|}{|z^k|} \; ; \quad 1 \geq \frac{|f(z^2)|}{|z^{2k}|} \geq 0. \quad \text{Hence}$$

$$|a_k| = \lim_{z \mapsto 0} \frac{|f(z)|}{|z|^k} \leq \frac{|f(z^2)|}{|z|^k} \leq |z^k| \longmapsto 0$$

Thus $a_k = 0$.