Math 742 Complex Variables Exam 3

The following problems were featured on qualifying exams in complex analysis at the graduate center. Submit your work on any 5.

1. Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial with coefficients in $\mathbb{C}$. Show that for all but finitely many $w \in \mathbb{C}$, $f(z) - w$ has n distinct roots in $\mathbb{C}$.

2. Let $f: \mathbb{C} \to \mathbb{C}$ be entire and suppose $f(S^1) = S^1$, where S^1 is the unit circle. Show that $f(z) = \alpha z^d$, for some $d \in \mathbb{N}$ and $|\alpha| = 1$.

3. Show that for any holomorphic function $f: \mathbb{D} \to \mathbb{D}$,

$$\left|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)}\right| \leq \left|\frac{z_1 - z_2}{1 - \overline{z_1} z_2}\right|$$

for all z_1, z_2 in the unit disc $\mathbb{D}$. Study the case of equality.

4. Show that for any holomorphic function $f: \mathbb{D} \to \mathbb{D}$

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$$

for all z in the unit disc $\mathbb{D}$. Study the case of equality.

5. If $f(z)$ is holomorphic and $Imf(z) \geq 0$ whenever $Im(z) > 0$, show that

$$\frac{|f(z) - f(z_0)|}{\left|f(z) - \overline{f(z_0)}\right|} \leq \frac{|z - z_0|}{|z - \overline{z_0}|}$$

and

$$\frac{|f'(z)|}{Imf(z)} \leq \frac{1}{Im(z)}$$

6. Suppose f is a holomorphic automorphism of the unit disc $\mathbb{D}$ such that f has two fixed points. Show that f must be the identity.

7. Let $\mathbb{P}$ denote the right half-plane $\{z: Re(z) > 0\}$. If $f: \mathbb{P} \to \mathbb{P}$ is holomorphic and $f(1) = 1$ show that

   (i) $|f'(1)| \leq 1$ and

   (ii) $\left|\frac{f(z)-1}{f(z)+1}\right| \leq \left|\frac{z-1}{z+1}\right|$ for all $z \in \mathbb{P}$.

8. Does there exist a holomorphic function $f: \mathbb{D} \to \mathbb{D}$ such that $f\left(\frac{1}{2}\right) = \frac{3}{4}$ and $f'\left(\frac{1}{2}\right) = \frac{2}{3}$?

9. Is there a holomorphic function $f: \mathbb{D} \to \mathbb{D}$ such that $f(0) = 1/2$ and $f'(0) = 3/4$? If so, find such an f. Is it unique?

10. Suppose $f: \mathbb{D} \to \mathbb{D}$ is holomorphic. Show that

$$\frac{|f(0)| - |z|}{1 - |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)||z|}$$

for all $|z| < 1$.

11. Suppose $f: \mathbb{D} \to \mathbb{D}$ is holomorphic and $|f(z^2)| \geq |f(z)|$ for all $z \in \mathbb{D}$. Show that f is a constant.