Math 352 HW. # 2

Homework problems are taken from "Real Analysis" by N. L. Carothers. The problems are color coded to indicate level of difficulty. The color green indicates an elementary problem, which you should be able to solve effortlessly. Yellow means that the problem is somewhat harder. Red indicates that the problem is hard. You should attempt the hard problems especially.

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. Prove that $\mathbb{N}$ (with its usual metric) is homeomorphic to $\left\{\frac{1}{n}:\ n \geq 1\right\}$ (with its usual metric).

2. Show that every metric space is homeomorhpic to one of finite diameter. [Hint: Every metric is equivalent to a bounded metric.]

3. Suppose that we are given a point x and a sequence $\{x_n\}$ in a metric space *M*, and suppose that $f(x_n) \to f(x)$ for every continuous real-valued function *f* on *M*. Prove that $x_n \to x$.

4. If $A \subset B \subset M$, and if *B* is totally bounded, show that *A* is totally bounded.

5. Show that *A* is totally bounded if and only if *A* can be covered by finitely many *closed* sets of diameter at most ε for $\varepsilon > 0$.

6. Prove that *A* is totally bounded if and only if $\overline{A}$ is totally bounded.

7. If *A* is *not* totally bounded, show that *A* has an infinite subset *B* that is homeomorphic to a discrete space.

8. Give an example of a closed bounded subset of ℓ_∞ that is not totally bounded.

9. Let *A* be a subset of an arbitrary metric space (M, d). If (*A*, d) is complete, show that A is closed in M.

10. Show that $\mathbb{R}$ endowed with the metric $\rho(x,\ y) = \left|\tan^{-1}(x) - \tan^{-1}(y)\right|$ is *not* complete. How about if we try $\tau(x,\ y) = \left|x^3 - y^3\right|$?

11. Prove or disprove: If M is complete and $f:(M,\ d)\to(N,\ \rho)$ is continuous, then *f*(M) is complete.

12. Given metric spaces M and N, show that M × N is complete if and only if both M and N are complete.

13. Let S denote the vector space of all finitely nonzero real sequences; that is, $x=\{x_n\}\in S$ if $x_n=0$ for all but finitely many n. Show that S is *not* complete under the sup norm $\|x\|_\infty=\sup_n|x_n|$.

14. Prove that C_0, the space of sequences that converge to 0, is complete by showing that C_0 is closed in ℓ_∞. [Hint: If $\{f_n\}$ is a sequence in C_0 converging to $f\in\ell_\infty$, note that $|f(k)|\le|f(k)-f_n(k)|+|f_n(k)|$. Now choose *n* so that the $|f(k)-f_n(k)|$ are small *independent* of *k*.]