Math 352 HW. # 1

Homework problems are taken from "Real Analysis" by N. L. Carothers. The problems are color coded to indicate level of difficulty. The color green indicates an elementary problem, which you should be able to solve effortlessly. Yellow means that the problem is somewhat harder. Red indicates that the problem is hard. You should attempt the hard problems especially.

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. If E is a connected subset of M, and if A and B are disjoint open sets in M with $E \subset A \cup B$, prove that either $E \subset A$ or $E \subset B$.

2. Prove that E is disconnected if and only if there exist nonempty sets A and B in M satisfying $A \cap \overline{B} = \varnothing$, $B \cap \overline{A} = \varnothing$, and $E = A \cup B$.

3. If E and F are connected subsets of M with $E \cap F \neq \varnothing$, show that $E \cup F$ is connected.

4. If every pair of points in M is contained in some connected set, show that M is itself connected.

5. If E and F are nonempty subsets of M, and if $E \cup F$ is connected, show that $\overline{E} \cap \overline{F} \neq \varnothing$.

6. If $A \subset B \subset \overline{A} \subset M$, and if A is connected, show that B is connected. In particular, $\overline{A}$ is connected.

7. True or false? If $A \subset B \subset C \subset M$, where A and C are connected, then B is connected.

8. If M is connected and has at least two points, show that M is uncountable. [Hint: Find a nonconstant, continuous, real-valued function on M.]

9. If $f : [a,\ b] \to [a,\ b]$ is continuous, show that f has a fixed point; that is, show that there is some point x in [a, b] with f(x) = x.

10. Let $f : [a,\ b] \to R$ be continuous with f(0) = f(2). Show that there is some x in [0, 1] such that f(x) = f(x + 1).

11. Prove that there does not exist a continuous function $f : R \to R$ satisfying $f(Q) \subset R \setminus Q$ and $f(R \setminus Q) \subset Q$.

12. Let A and B be closed subsets of M, and suppose that both $A \cup B$ and $A \cap B$ are connected. Prove that A and B are connected.

13.

(a) Give an example of a continuous function having a connected range but a disconnected domain.

(b) Let $D \subset R$ and let $f : D \to R$ be continuous. Prove that D is connected if $\{(x, \ f(x)) : x \in D\}$, the graph of *f* is a connected subset of R^2.

14. Let $f : [0, \ 1] \to R$ be defined by $f(x) = Sin(1/x)$ for $x \neq 0$ and *f*(0) = 0. Show that although *f* is not continuous, the graph of *f* is a connected subset of R^2.