Solutions to Hand-In Assignment 3

1. Which of the following are metric functions on $(0, \infty)$? Write simply "metric" or "not metric".

   a) $d(x, \; y) = \left| \frac{1}{x^4} - \frac{1}{y^4} \right|$ [1 pt]

   Solution: metric

   b) $d(x, \; y) = |x - 3y|$ [1pt]

   Solution: not metric

   c) $d(x, \; y) = \sqrt{|x-y|} + \frac{|x-y|}{1+|x-y|}$ [1 pt]

   Solution: metric

   d) $d(x, \; y) = \tan^{-1}|x-y|$ [1 pt]

   Solution: metric

   e) $d(x, \; y) = \min\left\{ |x-y|^{3/4}, \; 2 \right\}$ [1 pt]

   Solution: metric

   Which of the following are metric functions on $(0, \infty) \times (0, \infty)$? Write simply "metric" or "not metric".

   f) $d((x, \; y), \; (w, \; z)) = \sqrt{\left| \frac{1}{x^4} - \frac{1}{w^4} \right|^2 + \left| \frac{1}{y^4} - \frac{1}{z^4} \right|^2}$ [1 pt]

   Solution: metric

   g) $d((x, \; y), \; (w, \; z)) = |x - 3w| + |y - z|$ [1 pt]

   Solution: not metric

   h) $d((x, \; y), \; (w, \; z)) = \sqrt{|x-w|} + \frac{|y-z|}{1+|y-z|}$ [1 pt]

   Solution: metric

i) $d((x,\ y),\ (w,\ z)) = \tan^{-1}\left(\sqrt{|x-w|^2 + |y-z|^2}\right)$ [1 pt]

Solution: metric

j) $d((x,\ y),\ (w,\ z)) = \min\{|x-w|^{3/4},\ 2\} + \min\{|y-z|^{1/4},\ 1\}$ [1 pt]

Solution: metric

2. Let M = (0, ∞) be supplied with the metric function $d(x,\ y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ and let $\{n\}_{n=1}^{\infty}$ be a sequence of positive integers.
   a) Is the sequence $\{n\}_{n=1}^{\infty}$ a Cauchy sequence in (M, d)? Justify your answer.

Solution: The sequence $\{n\}_{n=1}^{\infty}$ is Cauchy in (M, d). To see this, observe that $d(n,\ m) = \left|\frac{1}{n} - \frac{1}{m}\right| \le \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| \to 0+0$ as $m, n \to \infty$.

[6 pts]

   b) Does the sequence $\{n\}_{n=1}^{\infty}$ converge in (M, d)? Justify your answer.

[4 pts]

Solution: The sequence $\{n\}_{n=1}^{\infty}$ does not converge in (M, d): If the sequence were convergent, it would have to converge to an element $x \in$ (M, d). But $d(n,\ x) = \left|\frac{1}{n} - \frac{1}{x}\right| \to \left|\frac{1}{x}\right| \ne 0$ as $n \to \infty$.

3. True or false? $\left|\tan^{-1}|x| - \tan^{-1}|y|\right| \le \tan^{-1}|x-y|$ Justify your answer. [Hint: look at HW # 3, problem 1] [10 pts]

Solution: True. The function $d(x,\ y) = \tan^{-1}|x-y|$ is a metric on **R**. Therefore $|d(x,\ 0) - d(y,\ 0)| = \left|\tan^{-1}|x| - \tan^{-1}|y|\right| \le d(x,\ y) = \tan^{-1}|x-y|$.

4. Let (ℝ, d) be a metric space with the metric function $d(x,\ y) = \frac{|x-y|}{1+|x-y|}$. Calculate diam(0, ∞). [10 pts]

Solution: diam(0, ∞) = $\lim_{|x-y|\to\infty} \frac{|x-y|}{1+|x-y|} = 1$.