Solutions to Hand-In Assignment 1

Let A, $B \subset \mathbb{R}$ be nonempty.

1. Define $A + B = \{x + y : x \in A \text{ and } y \in B\}$.
   Compute sup ($A + B$) in terms of sup (A) and sup (B). Repeat exercise for inf ($A + B$). Justify your answer. [10 pts]

   Solution: Let z ∈ A + B. Then z = x + y for some choice of x ∈ A and y ∈ B and since x ≤ sup (A) and y ≤ sup (B) we must have z = x + y ≤ sup (A) + sup (B). In particular, sup (A) + sup (B) is an upper bound of A + B. Therefore, sup ($A + B$) ≤ sup (A) + sup (B).
   Note that sup (A) + sup (B) is the least upper bound if and only if it is an upper bound and *no smaller number* can be an upper bound. Now sup (A) + sup (B) – ϵ = [sup (A) – $\epsilon/2$] + [sup (B) – $\epsilon/2$] < a + b ∈ A + B, because sup (A) – $\epsilon/2$ is *not* an upper bound of A and sup (B) – $\epsilon/2$ is *not* an upper bound of B. We may therefore conclude that

   $$\sup (A + B) = \sup (A) + \sup (B).$$

   An analogous argument may be used to derive the formula inf (A + B) = inf (A) + inf (B). For a quicker proof, recall that inf (A) = -sup (-A). Therefore, inf (A + B) = -sup (-A + [-B]) = -sup (-A) + [-sup (-B)] = inf (A) + inf (B).

2. Let $c > 0$. Define $cA = \{c\,x : x \in A\}$. Compute sup (cA) in terms of sup (A). What happens if $c < 0$? Repeat exercise for inf (cA). [10 pts]

   Solution: Let z ∈ cA. Then z = c x for some x ∈ A. Since c > 0, z = c x < c sup (A) . In particular, c sup (A) is an upper bound for cA and sup (cA) ≤ c sup (A).
   To check whether c sup (A) is the least upper bound of cA, consider c sup (A) – ϵ = c (sup (A) – ϵ/c) < c x for some choice of x ∈ A. Therefore c sup (A) cannot be an upper bound and we may conclude that

   $$\sup (cA) = c \sup (A).$$

   If c < 0, c = -|c| and sup (cA) = sup (-|c|A) = |c| sup (-A) = - |c| inf (A) = c inf (A).

   The situation with inf (cA) is similar and could be conveniently solved by appealing to the identity inf (A) = -sup (-A). In particular, if c > 0, we have inf (cA) = -sup (-cA) = -c sup (-A) = c inf (A). You should also check that when c < 0, inf (c A) = c sup (A).

3. Define $AB = \{xy: x \epsilon A \text{ and } y \epsilon B\}$. Assuming that the elements of A and the elements of B are nonnegative, compute sup (AB) in terms of sup (A) and sup (B). Is your answer still true if we drop the assumption that A and B are nonnegative? [10 pts]

Solution: We break the problem into 3 cases:

Case 1: A or B is the singleton {0}. Assume without loss of generality that A = {0}. Then AB = {0} and so 0 = sup (AB) = sup (A) sup (B).

Case 2: A or B is unbounded and each set contains at least 2 elements. Assume without loss of generality that A is unbounded. Since the elements in A are nonnegative, we must have sup (A) = ∞. Let $x \in B$ be any element other than 0. Then sup (A) sup (B) $\geq (\infty)(x) = \infty$. On the other hand, the set AB contains real numbers of the form a x where a is arbitrarily large. Therefore sup (AB) = ∞ as well.

Case 3: A and B are bounded sets, each containing at least 2 elements. First observe that if $z \in$ AB, z = ab $\leq$ sup (A) sup (B). In particular, sup (AB) $\leq$ sup (A) sup (B), because sup (A) sup (B) is an upper bound of AB. To get the reverse inequality, we show that no upper bound of AB can be smaller than sup (A) sup (B). For that purpose, consider sup (A) sup (B) – ϵ, where $0 < \epsilon$. Notice that there is a $0 < \delta < 1$ such that

$$\sup(A)\sup(B) - \epsilon < [\sup(A) - \delta]\,[\sup(B) - \delta] < \sup(A)\sup(B). \qquad (1)$$

To show this, observe that the above inequality holds if and only if $-\epsilon < -\delta(\sup(A) + \sup(B) - \delta)$. Multiplying by –1 yields $\epsilon > \delta(\sup(A) + \sup(B) - \delta)$. Set y = sup (A) + sup (B) + 1 and apply the Archimedean property of real numbers to find a positive integer n such that n ϵ > y. Then $\epsilon > (1/n)\,y = (1/n)(\sup(A) + \sup(B) + 1) > (1/n)(\sup(A) + \sup(B) - (1/n))$. This shows that $\delta = 1/n$ is the desired number.

Since there is some $a \in$ A and some b $\in$ B such that $[\sup(A) - \delta]\,[\sup(B) - \delta] < ab$, inequality (1) clearly implies that sup (A) sup (B) – ϵ is *not* an upper bound of AB. We may therefore conclude that

$$\sup(AB) = \sup(A)\sup(B).$$

The identity sup (AB) = sup (A) sup (B) *does not have to hold* for arbitrary subsets A, B of **R**. For instance, if A = (-1, 0) and B = (-2, 0) then sup(AB) = 2 while sup(A) sup (B) = 0.

4. Suppose $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are real valued functions. Define $f(A) \oplus g(A) = \{f(x) + g(x): x \in A\}$ and $f(A) + g(A) = \{f(x) + g(y): x, y \in A\}$. What is the relationship between sup ($f(A) \oplus g(A)$) and sup ($f(A) + g(A)$)? Repeat exercise for inf ($f(A) \oplus g(A)$) [10 pts]

   Solution: Clearly $\{f(x) + g(x): x \in A\} \subset \{f(x) + g(y): x, y \in A\}$ and therefore sup ($f(A) \oplus g(A)$) ≤ sup ($f(A) + g(A)$). Note that by the work done in exercise 1, sup ($f(A) + g(A)$) = sup f (A) + sup g (A).
   The inequality sup ($f(A) \oplus g(A)$) ≤ sup ($f(A) + g(A)$) may be strict. Consider the case when A = [1, 2], $f(x) = x^2$, and $g(x) = -x^2$. Then sup ($f(A) \oplus g(A)$) = sup {0} = 0, while sup ($f(A) + g(A)$) = sup f (A) + sup g (A) = 4 –1 = 3.

   The comparison of inf ($f(A) \oplus g(A)$) and inf ($f(A) + g(A)$) is similar. We always have inf ($f(A) + g(A)$) ≤ inf ($f(A) \oplus g(A)$) and this inequality may be strict.