Solutions to HW#4

1. To prove that $C_r(x)$ is closed, it suffices to show that $[C_r(x)]^c$ is open. Let $y \in [C_r(x)]^c$. Then $d(x,y) = \rho > r$. Let $\epsilon = \rho - r > 0$. We now show that $B_\epsilon(y) \subset [C_r(x)]^c$. With that in mind, pick $z \in B_\epsilon(y)$. Then

$$d(z,x) > |d(z,y) - d(x,y)| = |\epsilon - \rho| = r$$

which implies that $z \in [C_r(x)]^c$. Thus $B_\epsilon(y) \subset [C_r(x)]^c$, which proves that $[C_r(x)]^c$ is open and therefore $C_r(x)$ is closed.

2. Let $\mathcal{O}_A = \{\mathcal{U} : \mathcal{U} \subseteq A \text{ \& } \mathcal{U} \text{ open in } M\}$ (We are here assuming that $A \subset (M,d)$)

Then $A^o = \bigcup_{\mathcal{U}:\mathcal{U}\in\mathcal{O}_A} \mathcal{U}$. Clearly $A^o \subseteq A$ by its definition.

If $A^o = A$, then A is the union of open sets, which means that A is open. If A is open then $A \in \mathcal{O}_A$ and for any $\mathcal{U} \in \mathcal{O}_A$, $\mathcal{U} \subseteq A$. Thus $\bigcup_{\mathcal{U}:\mathcal{U}\in\mathcal{O}_A} \mathcal{U} \subseteq A \subseteq \bigcup_{\mathcal{U}:\mathcal{U}\in\mathcal{O}_A} \mathcal{U} = A^o$. Hence $A^o = A$ if and only if A is open.

We now verify that A is closed if and only if $A = \overline{A}$. Define $\mathcal{L}_A = \{F : A \subseteq F \text{ \& } F \text{ is closed in } M\}$. Then $\overline{A} = \bigcap_{F:F\in\mathcal{L}_A} F$. Clearly $A \subseteq \overline{A}$ by its definition. If $A = \overline{A}$, then A is equal to the intersection of closed sets, which is closed. This implies that A is closed. On the other hand, if A is closed then $A \in \mathcal{L}_A$ and $\overline{A} = \bigcap_{F:F\in\mathcal{L}_A} F \subseteq A$

Since A is always a subset of $\bar{A}$, we get $\bar{A} \subseteq A \subseteq \bar{A}$, which implies that $A = \bar{A}$.

3. Let $E \subset \mathbb{R}$ be nonempty and bounded. Set $\alpha = \sup(E)$. Then for any $\epsilon > 0$, $B_\epsilon(\alpha) = (\alpha - \epsilon, \alpha + \epsilon)$ contains some $x \in E$ in the segment $(\alpha - \epsilon, \alpha) \subset B_\epsilon(\alpha)$ (Why?). In other words, α is a limit point of E. If $E \subset F$, where F is a closed set, α is also a limit point of F. Therefore $\alpha \in F$ (why?). Since $E \subseteq \bar{E}$, $\alpha \in \bar{E}$, which shows that $\sup(E)$ is an element of $\bar{E}$. The proof that $\inf(E)$ is also an element of $\bar{E}$ is similar.

4. First observe that if $A \subset B$ then $\{d(a,b) : a, b \in A\} \subseteq \{d(a,b) : a, b \in B\}$. Therefore $\operatorname{diam}(A) = \sup\{d(a,b) : a, b \in A\} \leq \sup\{d(a,b) : a, b \in B\} = \operatorname{diam}(B)$. Since $A \subseteq \bar{A}$, it follows that $\operatorname{diam}(A) \leq \operatorname{diam}(\bar{A})$. Thus $\alpha = \operatorname{diam}(\bar{A})$ is an upper bound of $\{d(a,b) : a, b \in A\}$. To show that $\operatorname{diam}(A) = \operatorname{diam}(B)$ it therefore suffices to prove that α is the least upper bound of the set $\{d(a,b) : a, b \in A\}$. Let $\epsilon > 0$. Then $\alpha - \epsilon$ is not an upper bound of $\{d(a,b) : a, b \in \bar{A}\}$ and there are points $x, y \in \bar{A}$ such that $d(x,y) > \alpha - \epsilon/2$ (why?) Notice however that for any $x, y \in \bar{A}$ we have $a, b \in A$ such that $d(x,a) < \epsilon/4$ and $d(y,b) < \epsilon/4$. Therefore

$$d(x,y) \leq d(x,a) + d(a,b) + d(b,y) < \epsilon/2 + d(a,b).$$

Thus $\alpha - \epsilon/2 < d(x,y) < \epsilon/2 + d(a,b)$ or

$$\alpha - \epsilon < d(a,b).$$

This means that $\alpha - \epsilon$ is also not an upper bound of $\{d(a,b) : a, b \in A\}$

Hence $\alpha = \sup\{d(a,b) : a, b \in A\}$ and $\operatorname{diam}(A) = \operatorname{diam}(\overline{A})$ as desired.

5. Recall that $B \subseteq \overline{B}$ for any set B. If $A \subset B$, then $A \subset B \subset \overline{B}$, which means that $\overline{B} \in \mathcal{L}_A = \{F : A \subset F \text{ \& } F \text{ is closed}\}$.
Hence $\overline{A} = \bigcap_{F : F \in \mathcal{L}_A} F \subseteq \overline{B}$, from which $\overline{A} \subseteq \overline{B}$ follows.

Note that $\overline{A} \subset \overline{B}$ does not imply $A \subset B$. In fact, $A \cap B$ could be empty. For instance, if $A = \mathbb{Q} \cap [0,1]$ and $B = \mathbb{R} \setminus \mathbb{Q} \cap [0,2]$ then $\overline{A} = [0,1] \subset [0,2] = \overline{B}$, but clearly $A \cap B = \emptyset$.

6. Let $A, B \subset M$. Observe that $\overline{A}$ is the smallest closed set that contains A. That is, if $A \subset F$ and F is closed, then $A \subseteq \overline{A} \subseteq F$.
Notice that $A, B \subset A \cup B \subseteq \overline{A \cup B}$. Since $\overline{A \cup B}$ is closed, $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$ by the above remark. Hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Similarly, since the union of two closed sets is again closed, we have $A, B \subseteq \overline{A} \cup \overline{B}$, which implies that $A \cup B \subseteq \overline{A} \cup \overline{B}$ from which $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ follows. We have shown that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
To prove that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$, notice that $A \cap B \subset \overline{A} \cap \overline{B}$ and since $\overline{A} \cap \overline{B}$ is closed, we have $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.
This time, equality does not always occur; if $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$ then $\overline{A \cap B} = \overline{\emptyset} = \emptyset$, whereas $\overline{A} \cap \overline{B} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$.

7. Observe that $A^\circ \cup B^\circ$ is always a subset of $(A \cup B)^\circ$ (why?), but equality does not always occur; let $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$. Then $A^\circ \cup B^\circ = \emptyset \cup \emptyset = \emptyset$ but $(A \cup B)^\circ = \mathbb{R}^\circ = \mathbb{R}$.

8. Observe that $x \notin \overline{A}$ if and only if $x \notin A$ and x is an isolated point of A. That is $[\overline{A}]^c = \bigcup_{x : B_\epsilon(x) \subset A^c} B_\epsilon(x)$.
A bit of thought should convince you that $[\overline{A}]^c = \text{int}(A^c)$.
Hence $\overline{A} = [\text{int}(A^c)]^c$ as desired.
To show that $A^\circ = [cl(A^c)]^c$, set $B = A^c$. Then
$cl(A^c) = \overline{B} = [\text{int}(B^c)]^c = [A^\circ]^c$ by the result obtained above.
Hence $[cl(A^c)]^c = A^\circ$.

9. Suppose that $A \subseteq \mathbb{R}$ is a nonempty, proper, open subset of $\mathbb{R}$.
We will show that A cannot be closed.
Let $x \in A$. Set $b = \sup\{y : [x, y) \subset A\}$ and $a = \inf\{z : (z, x] \subset A\}$.
Since A is a proper subset, either a or b must be a finite number (why?). Assume, without loss of generality, that $b < +\infty$. Since A is open, $b \notin A$ (Otherwise $b \in (b-\epsilon, b+\epsilon) \subset A$ for some $\epsilon > 0$, which would imply that $(a, b+\epsilon) \subset A$, contrary to our choice of b.) Hence $b \in A^c$ — a closed set. If A were clopen, A^c would have also been an open set. But this is impossible because for every $\epsilon > 0$, $B_\epsilon(b) \cap A \neq \emptyset$. Which implies that A^c does not contain an entire neighborhood of b.

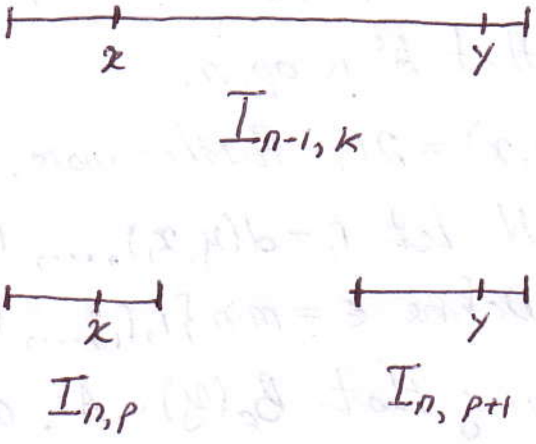
10. Suppose x is a limit point of $A \subset M$, let $B_r(x)$ be any neighborhood of x. Then, by our hypothesis on x, $B_r(x) \setminus \{x\} \cap A \neq \emptyset$. If $B_r(x) \setminus \{x\}$ were to contain only finitely many points $a_1, \ldots, a_n$ of A, we could set $\epsilon_1 = d(x, a_1), \ldots, \epsilon_n = d(x, a_n)$. Then each $\epsilon_i > 0$ (why?). Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. Then $B_\epsilon(x) \subset B_r(x)$ and $B_\epsilon(x) \setminus \{x\} \cap A = \emptyset$, contradicting the hypothesis that x is a limit point. Thus, each neighborhood of x must contain infinitely many

points of A.

11. Suppose $x_n \xrightarrow{d} x$ and $A = \{x\} \cup \{x_n : n \geq 1\}$. We will show that A is closed by proving that A^c is open.
Pick $y \in A^c$, then $d(y,x) = 2r$. Furthermore, for some $N \in \mathbb{N}$, $x_n \in B_r(x)$ for all $n \geq N$. Let $r_1 = d(y,x_1), \ldots, r_N = d(y,x_N)$. Then $r_1, \ldots, r_N > 0$ (why?) Define $\epsilon = \min\{r, r_1, \ldots, r_N\}$. Then $B_\epsilon(y) \cap A = \emptyset$, implying that $B_\epsilon(y) \subset A^c$, as you should verify. We have thus shown that A^c is open and that, therefore, A is closed as desired.

12. Recall that each point of Δ can be written using only the digits 0 and 2 in ternary (base 3) decimal expansion. Any number of the form $0.a_1a_2\ldots a_n11$ has only one other form, namely $0.a_1a_2\ldots a_n10222\ldots$ (why?). Hence it is clear that a number of the form $0.a_1\ldots a_n11$ cannot have the characteristic decimal expansion of elements in Δ. In particular, $0.a_1\ldots a_n11$ cannot be an element of Δ.

13. Every element in Δ is a limit of a sequence of nested closed subintervals. Let $x, y \in \Delta$ with $x < y$. Then $y - x = r$ and there is some n such that $3^{-(n-1)} \geq r$ while $3^{-n} < r$ (why?). This means that $x, y \in I_{n-1,k}$ where I_n is the "n^{th} level" and $I_{n,p}$ is the "n^{th} step" to the Cantor set. That is, $I_{n-1,k}$ is one of the 2^{n-1} subintervals of the $n-1$ Cantor level, of size $3^{-(n-1)}$
Since $y - x > 3^{-n}$, we see that $x \in I_{n,p}$ and $y \in I_{n,p+1}$ for some integer $1 \leq p \leq 2^n - 1$

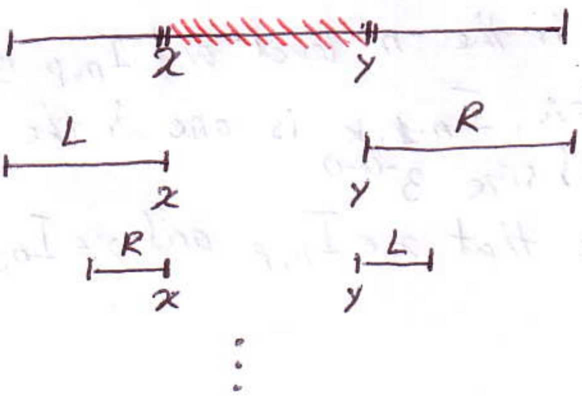


Pick any point z in the omitted interval segment. Then $z \notin \Delta$ and $x < z < y$, proving that Δ contains no open interval.
Since Δ is closed, we see that $\emptyset = \mathrm{int}(\Delta) = \mathrm{int}(\bar{\Delta})$, which establishes that Δ is nowhere dense.

14. Since Δ is the set of limits of sequences of left "L" and right "R" nested intervals, it is clear that two successive endpoints $x < y$ are of the form

$x \equiv A_1 A_2 \dots A_n L RRRR\dots$; $y \equiv A_1 A_2 \dots A_n R LLLL\dots$

(see fig. below).
This means that $x = 0.a_1 a_2 \dots a_n 0222\dots$ and $y = 0.a_1 a_2 \dots a_n 200\dots$ (base 3), where each a_i is either 0 or 2. Hence $x = 0.a_1 \dots a_n 1$ and $y = 0.a_1 a_2 \dots a_n 2$.



15. Each point $x \in \Delta$ is the limit of a sequence of nested, closed intervals $\{I_{n,k_n}\}$ with length $(I_{n,k_n}) = 3^{-n}$. That is, $\{x\} = \bigcap_{n=1}^{\infty} I_{n,k_n}$. Pick the right endpoints x_n of I_{n,k_n}. Then each $x_n \in \Delta$ and $x_n \mapsto x$. (If x is itself a right endpoint of some interval, use the left endpoints of I_{n,k_n})

Thus every point $x \in \Delta$ is a limit point of the endpoints of Δ. Furtheremore, since Δ is closed, we may conclude that Δ is perfect.

16. Suppose $x, y \in \Delta$ with $x < y$. Then

$x = 0.(2a_1)(2a_2)\ldots(2a_n)\ldots$ and $y = 0.(2b_1)(2b_2)\ldots(2b_n)\ldots$ where the a_i and b_i are either 0 or 1. Let n be the smallest integer for which $a_i < b_i$. Then $a_n = 0$ and $b_n = 1$. Also, note that $a_1 = b_1$, $a_2 = b_2, \ldots, a_{n-1} = b_{n-1}$.

$$\text{Now } f(x) = \sum_{k=1}^{\infty} \frac{a_k}{2^k} = \sum_{k=1}^{n-1} \frac{a_k}{2^k} + a_n + \sum_{k=n+1}^{\infty} \frac{a_k}{2^k}$$

$$= \sum_{k=1}^{n-1} \frac{b_k}{2^k} + \sum_{k=n+1}^{\infty} \frac{a_k}{2^k} \leq \sum_{k=1}^{n-1} \frac{b_k}{2^k} + \sum_{k=n+1}^{\infty} \frac{1}{2^k}$$

$$= \sum_{k=1}^{n-1} \frac{b_k}{2^k} + \frac{1}{2^n} = \sum_{k=1}^{n} \frac{b_k}{2^k} \leq \sum_{k=1}^{\infty} \frac{b_k}{2^k} = f(y)$$

Thus $f(x) \leq f(y)$ with equality holding if and only if $a_{n+1} = a_{n+2} = \ldots = 1$ and $b_{n+1} = b_{n+2} = \ldots = 0$. That is if and only if $x = 0.c_1c_2\ldots c_{n-1}1$ and $y = 0.c_1c_2\ldots c_{n-1}2$ where each c_i is either 0 or 2

That is, $f(x) = f(y)$ if and only if x and y are consequtive endpoints.