Solutions to HW#2

1. (i) Let $z: A \to A$ be defined by $z(x) = x$. Then clearly z is one-to-one and onto. Hence $A \sim A$.

(ii) Suppose $A \sim B$, then, by hypothesis, there is some $f: A \to B$ that is one-to-one and onto. Thus f is invertible and $f^{-1}: B \to A$ is also one-to-one and onto. In particular, $B \sim A$.

(iii) If $A \sim B$ and $B \sim C$, then, by hypothesis, there are some functions $f: A \to B$ and $g: B \to C$ that are one-to-one and onto. Clearly the function $h: A \to C$ defined by $h(x) = g(f(x))$ is one-to-one and onto. Hence $A \sim C$ as desired.

2. (a) First observe that for any countable sets A and B, $A \cup B$ is countable:

Arrange A and B into a matrix

$$\begin{array}{cccc}
A: & a_1 & a_2 & a_3 \ \cdots \\
 & (1) & (3) & (5) \\
B: & b_1 & b_2 & b_3 \ \cdots \\
 & (2) & (4) & (6)
\end{array}$$

and define $f: N \to A \cup B$ by $f(n) = \begin{cases} a_k & \text{if } n = 2k-1 \\ b_k & \text{if } n = 2k \end{cases}$

Observe that f is both 1-1 and onto. Hence $\mathbb{N} \sim A \cup B$ and the result follows.

We will now use induction to prove that $A_1 \cup \ldots \cup A_n$ is countable.

Clearly $A_1 \cup A_2$ is countable, so assume $A_1 \cup \ldots \cup A_{n-1}$ is countable. Let $A = A_1 \cup \ldots \cup A_{n-1}$ and $B = A_n$. Then, A and B are countable, from which it follows that

$A \cup B = A_1 \cup \ldots \cup A_n$ is countable.

(b) First, observe that if A and B are countable, then so is $A \times B$. To show this, arrange $A \times B$ in a matrix

$$\begin{array}{cccc} (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \ldots \\ (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & \ldots \\ (a_3, b_1) & (a_3, b_2) & (a_3, b_3) & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

Clearly $A \times B$ is equivalent to $\mathbb{N} \times \mathbb{N}$ (why?). Since $\mathbb{N} \times \mathbb{N}$ is equivalent to $\mathbb{N}$, it follows that $A \times B \sim \mathbb{N}$ and $A \times B$ is countable.

We will now use induction to prove that $A_1 \times \ldots \times A_n$ is countable.

Clearly $A_1 \times A_2$ is countable, so assume $A_1 \times \dots \times A_{n-1}$ is countable. Let $A = A_1 \times \dots \times A_{n-1}$ and $B = A_n$. Then A and B are countable, from which it follows that
$A \times B = (A_1 \times \dots \times A_{n-1}) \times A_n = A_1 \times \dots \times A_n$ is countable.

3. Suppose A is finite and $B \subset A$ is a proper subset of A. We can write $A = \{x_1, \dots, x_n\}$, where, without loss of generality, $x_1 \notin B$. Therefore $B \subset A \setminus \{x_1\}$.
It is easily seen from the pigeonhole principle that no map from $A \setminus \{x_1\}$ to A can be onto. Hence, no map from B to A is onto. In other words, A is not equivalent to any of its proper subsets.
Suppose, on the other hand, that A is infinite. Then, for some $x \in A$, $A \setminus \{x\}$ is infinite as well (Why?) Now, since every infinite set contains a countable subset, it follows that there is a sequence of distinct elements $\{y_n\}_{n=1}^{\infty} \subset A \setminus \{x\}$
Define $f : A \setminus \{x\} \to A$ by

$$f(y) = \begin{cases} x & \text{if } y = y_1 \\ y_{n-1} & \text{if } y = y_n;\ n \geq 2 \\ y & \text{otherwise} \end{cases}$$

you should verify that f is 1-1 and onto. We can therefore conclude that $A \sim A \setminus \{x\}$

4. Suppose A is infinite and B is countable. Then A contains a countably infinite subset that we can list in sequential order $\{a_n\}_{n=1}^{\infty}$. Since B is countable, we can list its elements in a sequence $\{b_n\}_{n=1}^{\infty}$

Define $f: A \to A \cup B$ by

$$f(x) = \begin{cases} a_n & \text{if } x = a_{2n} \\ b_n & \text{if } x = a_{2n-1} \\ x & \text{otherwise} \end{cases}$$

Clearly f is 1-1 and onto. Thus $A \sim A \cup B$. as desired.

5. Let A be countable and suppose $f: A \to B$ is onto. Then $A = f^{-1}(B) = \bigcup_{b:\, b \in B} A_b$ where $A_b = \{x \in A : f(x) = b\}$

Notice that since f is a function, $A_{b_1} \cap A_{b_2} = \emptyset$ whenever $b_1 \neq b_2$ (Why?) Furtheremore, since f is onto B, each $A_b \neq \emptyset$ (Why?). By the axiom of choice, we get a subset $K \subset A$ that contains one element from each A_b. Clearly the restricted function $f: K \to B$ is 1-1

and onto. Therefore $K \sim B$.
Since A is countable, every subset of A is at most countable. It follows that K and therefore B are at most countable.
Suppose $g: C \to A$ is one-to-one. Then $g: C \to g(C)$ is 1-1 and onto. It follows that $C \sim g(C) \subset A$. This implies that $g(C)$ is at most countable. Therefore C is at most countable.

6. To see that $\mathbb{R} \sim (0,1)$, observe that $f: \mathbb{R} \to (0,1)$ given by $f(x) = \dfrac{\tan^{-1}(x) + \frac{\pi}{2}}{\pi}$ is 1-1 and onto.

To prove that $(0,1) \sim [0,1]$, take $\{\frac{1}{n+1}\}_{n=1}^{\infty} \subset (0,1)$ and define $g: (0,1) \to [0,1]$ by

$$g(x) = \begin{cases} 0 & \text{if } x = \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{3} \\ \frac{1}{n-1} & \text{if } x = \frac{1}{n+1};\ n \geq 3 \\ x & \text{otherwise} \end{cases}$$

Then g is 1-1 and onto and the result $(0,1) \sim [0,1]$ follows.

Let $y \in (0,1)\times(0,1)$. Then $y=(y_1, y_2)$ where $y_1 = 0.a_1a_2a_3\ldots$ and $y_2 = 0.b_1b_2b_3\ldots$ are the unique infinite base 10 decimal representations of y_1 and y_2.

Define $f:(0,1)\to(0,1)\times(0,1)$ by

$$f(x) = (0.x_1x_3x_5\ldots x_{2n-1}\ldots,\ 0.x_2x_4x_6\ldots x_{2n}\ldots)$$

whenever $x = 0.x_1x_2x_3x_4x_5x_6\ldots x_{2n-1}x_{2n}\ldots$ is the unique infinite base 10 representation of x.

To see that f is onto, observe that if

$y = (0.a_1a_2a_3\ldots,\ 0.b_1b_2b_3\ldots)$, then

$x = 0.a_1b_1a_2b_2a_3b_3\ldots$ will be mapped by f to y (i.e. $f(x)=y$). And since we can recover the input x from the output y, we see that f is invertible and therefore 1-1.

We have thus shown that $(0,1)\sim(0,1)\times(0,1)$.

8. Let $L_\infty\{0,1\}$ be the set of all functions $f:\mathbb{N}\to\{0,1\}$. Since the set of all functions from $\mathbb{N}$ to any set M is just the set of sequences with range in M, we see that each f can be uniquely represented by a string of 0's and 1's. For example, $(0,0,1,1,0,0,1,1,\ldots)$ refers to the function $f\in L_\infty\{0,1\}$ defined by $f(1)=0$, $f(2)=0$, $f(3)=1$, etc. In general, $f(n)=1$ whenever $n=3k$ or $4k$ and $f(n)=0$

otherwise.

Notice that each $x \in (0,1)$ has a unique infinite base 2 decimal representation. In other words, we can write $x = 0.x_1 x_2 x_3 \ldots$ (base 2) where each $x_i = 0$ or 1.

Define $\varphi : (0,1) \to L_\infty\{0,1\}$ by $\varphi(x) = f_x$, where $f_x(n) = x_n$.

In other words, if $x = 0.x_1, x_2, x_3 \ldots$ (base 2), then $f_x = (x_1, x_2, x_3, \ldots)$.

Clearly φ is 1-1 and onto. Thus $(0,1) \sim L_\infty\{0,1\}$ as desired.

9. Define $L_A\{0,1\}$ to be the set of all functions $f : A \to \{0,1\}$ and let $\mathcal{P}(A)$ denote the power set of A.

Define $\varphi : \mathcal{P}(A) \to L_A\{0,1\}$ by $\varphi(S) = f_S$, where

$$f_S(x) = \begin{cases} 0 & \text{if } x \notin S \\ 1 & \text{if } x \in S \end{cases}$$

To see that φ is onto, pick any $f : A \to \{0,1\}$ and set $S = f^{-1}\{1\}$. Then clearly $\varphi(S) = f_S = f$. In particular, each $f \in L_A\{0,1\}$ can be identified with the set $f^{-1}\{1\}$.

To see that φ is 1-1, notice that if $\varphi(S) = \varphi(T)$, then $f_S = f_T$. In other words, $f_S(x) = f_T(x)$ for all $x \in A$. This means that $S = \{x \in A ; f_S(x) = 1\} = \{x \in A ; f_T(x) = 1\}$ $= T$.

We conclude that $\mathcal{P}(A) \sim L_A\{0,1\}$

10. let P_n be the n^{th} prime. Define $\mathbb{N}_{P_n} = \{m \in \mathbb{N} : m = P_n^k ; k \in \mathbb{N}\}$
Thus, for example $\mathbb{N}_{P_1} = \mathbb{N}_2 = \{2, 4, 8, 16, 32, \ldots\}$
Clearly each $\mathbb{N}_{P_n}$ is infinite and $\mathbb{N}_{P_n} \cap \mathbb{N}_{P_t} = \emptyset$ whenever $n \neq t$. (Why?) Thus the sequence of sets $\{\mathbb{N}_{P_n}\}_{n=1}^{\infty}$ consists of infinite pairwise disjoint subsets of $\mathbb{N}$.

11. The hint pretty much sums up the argument. Let $\{q_n\}$ be some list of all the rational numbers and $\mathcal{L}$ be a collection of pairwise disjoint, nonempty open intervals in $\mathbb{R}$. Then each $I \in \mathcal{L}$ contains some rational r_I that is not found in any other open interval $J \in \mathcal{L}$. (Why?)
Let $K = \{r_I \in \mathbb{Q} : r_I \in I\}$ be a collection of rational representatives of the intervals of $\mathcal{L}$, - one representative from each $I \in \mathcal{L}$. Then we can lable $I = I_1$ if $r_I = q_{n_1}$, where n_1 is the smallest integer $n \in \mathbb{N}$ such that $q_n = r_I$ for any $I \in \mathcal{L}$. More generally, $I = I_k$ if the rational chosen in I, r_I, is equal to q_{n_k}, where n_k is the k^{th} smallest integer $n \in \mathbb{N}$ for which q_n is among the elements of K.

12. Since $\mathbb{R} \sim (0,1)$, we might as well show that a collection $\{\mathbb{N}_\alpha\}_{\alpha \in (0,1)}$ exists for which $\mathbb{N}_\alpha$ is infinite for each $\alpha \in (0,1)$, but $\mathbb{N}_\alpha \cap \mathbb{N}_\beta$ is finite whenever $\alpha \neq \beta$. To do this, recall that each $\alpha \in (0,1)$ can be written uniquely as an infinite binary (base 2) decimal expansion $\alpha = 0.a_1 a_2 a_3 \ldots$ where each $a_i = 0$ or 1. Let $P_0 = 2$ and $P_1 = 3$

Define $\mathbb{N}_\alpha = \left\{ P_{a_1}, P_{a_1}^{P_{a_2}}, P_{a_1}^{P_{a_2}^{P_{a_3}}}, P_{a_1}^{P_{a_2}^{P_{a_3}^{P_{a_4}}}}, \ldots \right\}$

For example, if $\alpha = 0.010011\ldots$ then $\mathbb{N}_\alpha =$

$\left\{ 2, 2^3, 2^{3^2}, 2^{3^{2^2}}, 2^{3^{2^{2^3}}}, 2^{3^{2^{2^{3^3}}}}, \ldots \right\} = \{2, 8, 512, \ldots\}$

Observe that the n^{th} largest number of $\mathbb{N}_\alpha$ encodes the first n terms of the binary expansion of α. Since we are using the unique infinite representation of α in binary code, $\mathbb{N}_\alpha$ is infinite.

If $\alpha, \beta \in (0,1)$ with $\alpha \neq \beta$, then without loss of generality, $\alpha < \beta$ and $\alpha = 0.a_1 a_2 a_3 \ldots a_n \ldots$ and $\beta = 0.b_1 b_2 b_3 \ldots b_n \ldots$ will first be different in the n^{th} digit, where $n = 1, 2, 3 \ldots$ That is, $a_i = b_i$ for $i = 1, 2, 3, \ldots, n-1$, while $a_n = 0$ and $b_n = 1$

Clearly then $\mathbb{N}_\alpha \cap \mathbb{N}_\beta$ have $n-1$ terms in common.