H.W. #1

1. Suppose $r \in \mathbb{Q}$ and $x \notin \mathbb{Q}$. If $x+r \in \mathbb{Q}$ then $x+r=s$, for some rational number s. This implies that $x = s-r \in \mathbb{Q}$, which contradicts our hypothesis about x. The proof that rx is irrational for every $r \in \mathbb{Q}$ $(r \neq 0)$ is similar.

2. Suppose $\left(\frac{m}{n}\right)^2 = 12$ and $\gcd(m,n)=1$.
Then $m^2 = 4(3n^2)$, implying that $3 \mid m^2$. Since 3 is prime, $3 \mid m$. In particular, $m = 3k$ for some integer k.
Thus $m^2 = 9k^2 = 4(3n^2)$ or, equivalently, $3k^2 = 4n^2$.
Since $3 \mid 4n^2$ and $3 \nmid 4$, it follows that $3 \mid n^2$ and, therefore, $3 \mid n$. It follows that $n = 3p$ for some $p \in \mathbb{Z}$. This is a contradiction, because $\gcd(m,n) = \gcd(3k, 3p) \geq 3 > 1$ violates the assumption $\gcd(m,n)=1$.

3. Let $x \in E$ and suppose α is a lower bound of E and β is an upper bound of E. Then $\alpha \leq x \leq \beta$.
In particular $\alpha \leq \beta$.

4. Suppose $A \subseteq \mathbb{R}$ is bounded below by α. That is, $\alpha \leq x$ for all $x \in A$. Define $-A = \{-x : x \in A\}$. Then $-A$ is bounded above by $-\alpha$ (why?)

Let $\beta = \sup(-A)$. Notice that $-x \leq \beta$ for all $-x \in -A$. This means that $-\beta \leq x$ for all $x \in A$. In particular, $-\beta$ is a lower bound of A. Let $\epsilon > 0$, then $\beta - \epsilon$ is not an upper bound of $-A$ (why?); there exists some $-x \in -A$ such that $\beta - \epsilon < -x \leq \beta$. It follows that $-\beta + \epsilon > x \geq -\beta$, which tells us that $-\beta + \epsilon$ is not a lower bound. We conclude that $-\beta$ is the greatest lower bound of A. That is $\inf A = -\beta = -\sup(-A)$.

5. (a) Let $r = \frac{m}{n} = \frac{p}{q}$. Then $mq = np$ and

$$\left((b^m)^{1/n}\right)^{nq} = b^{mq} = b^{np} = \left((b^p)^{1/q}\right)^{nq}$$

Since roots are unique, it follows that $(b^m)^{1/n} = (b^p)^{1/q}$. Hence it makes sense to define $b^r = (b^m)^{1/n}$

(b) Let $r = \frac{m}{n}$ and $s = \frac{c}{t}$. Then

$$(b^{r+s})^{nt} = b^{mt+nc} = b^{mt} b^{nc} = \left((b^{mt})^{1/nt}\right)^{nt}\left((b^{nc})^{1/nt}\right)^{nt} =$$

$$= \left(b^{\frac{m}{n}}\right)^{nt}\left(b^{\frac{c}{t}}\right)^{nt} = (b^r)^{nt}(b^s)^{nt} = (b^r b^s)^{nt}.$$

Since roots are unique, it follows that $b^{r+s} = b^r b^s$.

(c) Let $s, t \in \mathbb{Q}$ with $s < t$. Then $t - s = \frac{m}{n} > 0$. and

$$\left(\frac{b^t}{b^s}\right)^n = (b^{t-s})^n = \left(b^{\frac{m}{n}}\right)^n = b^m.$$ Since $b > 1$, it follows

that $b^m > b > 1$. We conclude that $(b^m)^{1/n} > 1$ (why?)
Thus $b^{t-s} > 1$ or, equivalently, $b^t > b^s$.

If we define $B(x) = \{b^t : t \in \mathbb{Q}\ \&\ t \leq x\}$, we obtain $b^r = \sup B(r)$, because for any $t < r$ $b^t < b^r$ and $b^r \in B(r)$.

(d) Let $b^p \in B(x)$ and $b^q \in B(y)$. Then $p, q \in \mathbb{Q}$ and $p < x$, $q < y$. It follows that $b^p b^q = b^{p+q} \in B(x+y)$.
Therefore $\sup B(x) \sup B(y) \leq \sup B(x+y)$.

Let $t < x+y$. Then $t - x < y$. Since the rationals are dense on the real number-line there exists $s \in \mathbb{Q}$ such that $t - x < s < y$. Since $t - s < x$, there exists some $r \in \mathbb{Q}$ such that $t - s < r < x$. Hence $t < r+s < x+s < x+y$ where $r < x$ and $s < y$. Thus $b^t < b^{r+s} = b^r b^s \leq$
$\leq \sup B(x) \sup B(y) = b^x b^y$. In other words, $b^{x+y} = \sup B(x+y)$
$\leq b^x b^y$.

Since $b^{x+y} \leq b^x b^y$ and $b^x b^y \leq b^{x+y}$, it follows that $b^{x+y} = b^x b^y$.

6. (a) $b^n - 1 = (b-1)(b^{n-1} + b^{n-2} + \ldots + 1) > (b-1)(1+1+\ldots+1) =$
$= (b-1)n$.

(b) Let $\alpha = b^{1/n}$. Then $\alpha^n - 1 > n(\alpha - 1)$ by part (a) (since $\alpha > 1$). Now $\alpha^n = b$ and the inequality can be expressed as $(b-1) > n(b^{1/n}-1)$

(c) If $t>1$ and $n > \frac{b-1}{t-1}$, then, by part (b), $n(t-1) > b-1 > n(b^{1/n}-1)$. The inequality $n(t-1) > n(b^{1/n}-1)$ is equivalent to $t > b^{1/n}$.

(d) Suppose $b^{\omega} < y$, then $yb^{-\omega} > 1$. Setting $t = yb^{-\omega}$ and $n > \frac{b-1}{t-1}$ yields $b^{1/n} < yb^{-\omega}$ (by part (c)). Therefore $b^{\omega}b^{1/n} = b^{\omega+1/n} < y$.

(e) Suppose $b^{\omega} > y$, then $y^{-1}b^{\omega} > 1$. Setting $t = y^{-1}b^{\omega}$ and $n > \frac{b-1}{t-1}$ yields $b^{1/n} < t = y^{-1}b^{\omega}$ (by part (c)). Thus $y < b^{\omega}b^{-1/n} = b^{\omega - 1/n}$.

(f) Let $A = \{\omega \in \mathbb{R} : b^{\omega} < y\}$ Then

i) $A \neq \emptyset$: If $y > 1$, set $n > \frac{b-1}{y-1}$ and use part (c) to conclude that $b^{1/n} < y$. In other words, if $y>1$, $1/n \in A$.
If $y = 1$, then $b^0 = 1 = y$. Hence $0 \in A$.
If $y < 1$, $\frac{1}{y} > 1$ and setting $m > \frac{b-1}{1/y - 1}$ yields $b^{1/m} < \frac{1}{y}$ by part (c). It follows that $y < b^{-1/m}$. In any case, A is not empty.

ii) A is bounded above: Define $B = \{b^n : n \in \mathbb{N}\}$. Then B does not have an upper bound; Assume that it does. Set $\sup B = S$. Since $b > 1$, $\frac{S}{b} < S$. In particular, $\frac{S}{b}$ is not an upper bound of B. There exists some $n \in \mathbb{N}$

such that $b^n > \frac{s}{b}$. But then $b^{n+1} > s$ which contradicts the assumption $s = \sup B$. It follows that B is not bounded above. This means that for some integer $k \in \mathbb{N}$, $b^k > y$. Since $w < k$ implies $b^w < b^k$ (and $b^w < b^k$ implies that $w < k$), we see that A is bounded above by k.

Let $x = \sup A$. We wish to show that $b^x = y$.

If $b^x < y$, part (d) implies that $b^{x+1/n} < y$ for some sufficiently large n. Thus $b^x < b^{x+1/n} < y$ and $x + 1/n \in A$ in contradiction to the assumption $x = \sup A$.

If $b^x > y$, part (e) implies that $b^{x-1/n} > y$ for some sufficiently large n. Thus $x - 1/n$ is an upper bound of A. (Why is this impossible?)

(g) If α and β satisfy $b^\alpha = b^\beta = y$ then $\alpha = \beta$. This follows from the fact that if $\alpha < \beta$ then there are rationals r, s that satisfy $\alpha < r < s < \beta$. Thus $b^\alpha < b^r < b^s < b^\beta$ by the work done in the previous problem.

It follows that an x satisfying $b^x = y$ is unique.

7. Suppose $x = \frac{a_1}{p} + \dots + \frac{a_n}{p^n}$. Then

$$x = \frac{a_1}{p} + \dots + \frac{a_{n-1}}{p^n} + \sum_{i=n+1}^{\infty} \frac{p-1}{p^i} \quad (\text{why?})$$

Let $0.b_1b_2\ldots b_n\ldots$ and $0.c_1c_2\ldots c_n\ldots$ be any two base p decimal expansions for x and suppose n is the first integer for which $b_i \neq c_i$. Then, without loss of generality, $b_1 = c_1, b_2 = c_2, \ldots, b_{n-1} = c_{n-1}, b_n < c_n$.

Thus,

$$0.b_1b_2\ldots b_n\ldots = \sum_{i=1}^{\infty} \frac{b_i}{p^i} \leq \sum_{i=1}^{n} \frac{b_i}{p^i} + \sum_{i=n+1}^{\infty} \frac{p-1}{p^i} =$$

$$= \frac{b_1}{p} + \frac{b_2}{p^2} + \ldots + \frac{b_n+1}{p^n} \leq \frac{c_1}{p} + \frac{c_2}{p^2} + \ldots + \frac{c_n}{p^n} \leq$$

$$\leq \sum_{i=1}^{\infty} \frac{c_i}{p^i} = 0.c_1c_2\ldots c_n\ldots$$

with equality if and only if $b_{n+i} = p-1$, $c_n = b_n + 1$, and $c_{n+i} = 0$ for all $i \geq 1$.

This means that if x has two decimal expansions, one of them must be finite. Hence if x does not have a finite decimal expansion (mod P), it's representation is unique.

8. If order is imposed on $\mathbb{C}$, then, for each $z \in \mathbb{C}$ $(z \neq 0)$ either $z > 0$ or $z < 0$.

Let $z = i$. By proposition 1.18 (d) $z^2 > 0$ for any $z \neq 0$. Thus $-1 = i^2 > 0$. However, since $1 = 1^2 > 0$ (by 1.18 (d)), it follows that both 1 and -1 are greater than 0. This violates proposition 1.18 (a). Thus $\mathbb{C}$ cannot be an ordered field.

9. The proof that the lexicographic order makes $\mathbb{C}$ into an ordered set is trivial. To see wether or not $\mathbb{C}$ is transformed into a set with the least-upper bound property. set $A = \{bi : b \in \mathbb{R}\}$. Then A is bounded above by any element $z \in \mathbb{C}$ for which $Re(z) > 0$. Observe also that if $z = a + bi$ with $a = Re(z) \leq 0$ then $\omega = (|b|+1)i \in A$ satisfies $\omega > z$

Although A is bounded above, A does not have a least-upper bound. To see this, suppose $\alpha + \beta i$ is an upper bound. Then $\alpha > 0$ and $\frac{\alpha}{2} + \beta i$ is also an upper-bound with $\frac{\alpha}{2} + \beta i < \alpha + \beta i$.

10. Let $z = a + bi$ and $\omega = u + vi$. Then $z^2 = \omega$ if and only if the equations

$$a^2 - b^2 = u \qquad (1)$$

$$2ab = v \qquad (2)$$

are satisfied. Write $b = \frac{v}{2a}$ and plug into first equation to obtain $a^2 - \frac{v^2}{4a^2} = u$. Multiply the last equation by a^2 to obtain $a^4 - a^2u - \frac{v^2}{4} = 0$. Then $a^2 = \frac{u + \sqrt{u^2 + v^2}}{2}$. Since $b^2 = u - a^2$ by equation (1),

$b^2 = \frac{-u+\sqrt{u^2+v^2}}{2}$. Therefore $a^2 = \frac{|w|+u}{2}$ and $b^2 = \frac{|w|-u}{2}$,
from which we obtain $a = \pm\left(\frac{|w|+u}{2}\right)^{1/2}$ and $b = \pm\left(\frac{|w|-u}{2}\right)^{1/2}$.
If $v>0$ then $2\left(\frac{|w|+u}{2}\right)^{1/2}\left(\frac{|w|-u}{2}\right)^{1/2} = 2\left(\frac{|w|^2-u^2}{4}\right)^{1/2} =$
$= 2\left(\frac{v^2}{4}\right)^{1/2} = |v| = v$.
Similarly, $2\left[-\left(\frac{|w|+u}{2}\right)^{1/2}\right]\left[-\left(\frac{|w|-u}{2}\right)^{1/2}\right] = v$ if $v>0$.
Thus $|a|+|b|i$ and $-(|a|+|b|i)$ are solutions to the equation $z^2=w$ in this case.
If $v<0$, then $-|a|+|b|i$ and $|a|-|b|i$ are solutions to $z^2=w$.
We see that if $w \neq 0$ the equation $z^2=w$ has at least two solutions. It can be shown that a polynomial equation of degree n can have at most n solutions. In particular, $z^2-w=0$ can have at most 2 solutions. Thus if $w\neq 0$, the equation $z^2=w$ has exactly two solutions.

11. Let $z\neq 0$ and set $r=|z|$ and $w=\frac{z}{|z|}$ then $|w|=1$ and $r>0$. Clearly $z=rw$. To see that z determines r and w uniquely, suppose $z=\rho u$, where $\rho>0$ and $|u|=1$. Then $|z|=|\rho u| = |\rho||u| = \rho$. But $|z|=r$. Hence $\rho=r$. Now $\frac{1}{r}z = \frac{1}{r}\rho u = \frac{1}{r}rw$. It follows that $u=w$.
Remark: If $z=0$, $z=rw$ when $r=0$ and $|w|=1$.

12. This follows by repeatedly applying theorem 1.33 (e).

13. Suppose $z\bar{z}=1$. Then $|1+z|^2+|1-z|^2 =$

$= (1+z)\overline{(1+z)} + (1-z)\overline{(1-z)} = (1+z)(1+\bar{z})+(1-z)(1-\bar{z}) =$

$= (1+\bar{z}+z+z\bar{z}) + (1-\bar{z}-z+z\bar{z}) = (2+\bar{z}+z)+(2-\bar{z}-z) =$

$= 4.$