Analysis – (roughly) the study of limits (in abstract settings).

- Real Analysis – Study of limits over real numbers
- Complex Analysis – Study of limits over complex numbers
- Analysis on Manifolds – Study of limits of multi-variable functions
- Fourier Analysis – Study of limits in connection to waves.

Limits cannot be thoroughly understood without a prior examination of the numbers over which the limit is taken. For example, the derivative is defined over real numbers and over complex numbers, but the mean-value thm is true over real numbers and not true over complex numbers.

Mean-Value thm: If $f(x)$ is differentiable, $\frac{f(b)-f(a)}{b-a} = f'(c)$ where $c \in (a,b)$.

The first quest of this course is to pin down the concept of real number.

Among other things, the pythagoreans are famous for the phrase "All is number." Have you wondered what this phrase means?

Origin of number

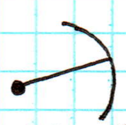
Modern mathematics evolved from the need to make precise measurements. One of the earliest branches of mathematics to develop was geometry. In building construction, a blue prints must be faithfully scaled up etc.

Tools: (1) Straight edge

(Stretched piece of rope).

(2) Unit edge (cut piece of rope that is identified as unit length).

(3) Compass



(stretched piece of rope that is fixed to the ground with one end)

Using these tools, it is easy to construct line segments and circles.

Ex. Construct a line segment of length 3-units

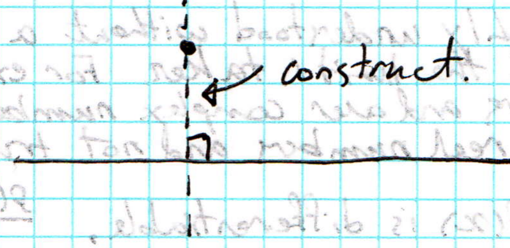
Solution:

The rules for constructing are very strict:

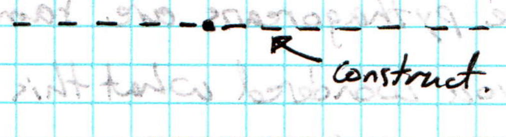
(1) A line / line segment may only be drawn between two previously constructed points (or by placing a previously constructed line segment at a point that lies on a line)

(2) A point can only be constructed by intersecting 2 lines, a line and a circle, or by intersecting two circles.

Problem 1: Given a line and a point not on the line, construct a perpendicular line through the point.



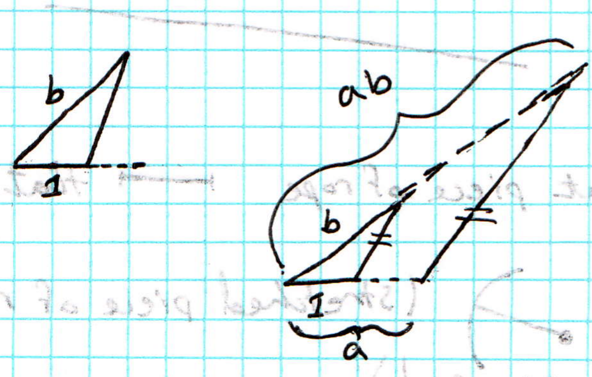
Problem 2: Given a line and a point not on the line, construct a ~~perpend~~ parallel line through the point.



Having the ability to construct parallel lines, the ancient geometers knew how to multiply one length times another

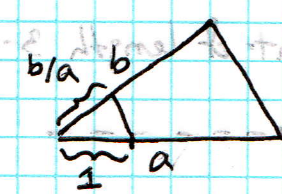
Ex. Multiply a times b

Solution:



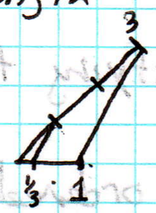
Ex. Divide b by a

Solution:



Ex. Trisect a unit length

Solution:

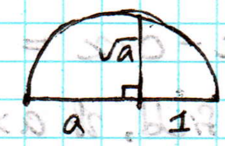


Do you see how every fraction $\frac{a}{b}$; $a, b \in \mathbb{N}$ can be constructed?

Using the ability to construct circles and lines the greek geometers knew how to take square roots.

Ex. Given length a, ~~find $\sqrt{a}$~~. construct $\sqrt{a}$. That is construct a number b such that $b \cdot b = a$.

Solution:



Circle of diameter $a+1$.

- Constructing ~~roots~~ square roots is related to the problem of doubling the area, / How to construct a square whose area is a.
- interestingly enough $\sqrt[3]{2}$ cannot be constructed.

Rational numbers

$\mathbb{Q} = \{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \}$; $\frac{m}{n} + \frac{p}{q} = \frac{qm + np}{nq}$, $\frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$

It is not difficult to verify that under the operation of addition & multiplication, as defined above, the rational numbers satisfy the following properties:

(1) $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbb{Q}$

(2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in \mathbb{Q}$

(3) $x+y = y+x$, & $x \cdot y = y \cdot x$

(4) $\exists$ unique element $0 \in \mathbb{Q}$ s.t. $x + 0 = x$ $\forall x \in \mathbb{Q}$

(5) $\exists$ unique element $1 \in \mathbb{Q}$ s.t. $x \cdot 1 = x$ $\forall x \in \mathbb{Q}$

(6) for each $x \in \mathbb{Q}$ $\exists$ unique $y \in \mathbb{Q}$ s.t. $x + y = 0$ (y denoted $-x$)

(7) for each $x \neq 0$ $\exists$ unique $y \in \mathbb{Q}$ s.t. $x \cdot y = 1$ (y denoted x^{-1})

(8) $x\cdot(y+z) = (x\cdot y) + (x\cdot z)$

Def: Any algebraic structure satisfying the above properties is called a field.

• The rational numbers are are ordered by magnitude; (e.g. $\frac{1}{4} < \frac{1}{3}$). and satisfy

(9) If $x > y$, then $x+z > y+z$

(10) If $x > y$, then, given $z > 0$, $x\cdot z > y\cdot z$

Def: Any field F satisfying (1)-(10) is called an ordered field.

Ex. Show that $0\cdot x = 0$ in any field.

Solution: $0\cdot x = (0+0)\cdot x = 0\cdot x + 0\cdot x$ Thus

$$0 = 0\cdot x - 0\cdot x = 0\cdot x + 0\cdot x - 0\cdot x = 0\cdot x.$$

Ex. Show that in any ordered field, if $a > 0$ then $-a < 0$

Solution: $-a = -a + 0 < -a + a = 0$

Ex. (a) Show that in any ordered field, if $a > 0$ and $b > 0$ then $ab > 0$

(b) $a(-b) < 0$

(c) $(-a)(-b) > 0$

Solution: (a) $ab > a\cdot 0 = 0$ by (10)

(b) $a(-b) + ab = a(-b+b) = 0$. Thus $a(-b) = -(ab)$

Since $ab > 0$, $-(ab) < 0$.

(c) $(-a)(-b) + a(-b) = (-a+a)(-b) = 0$. Thus

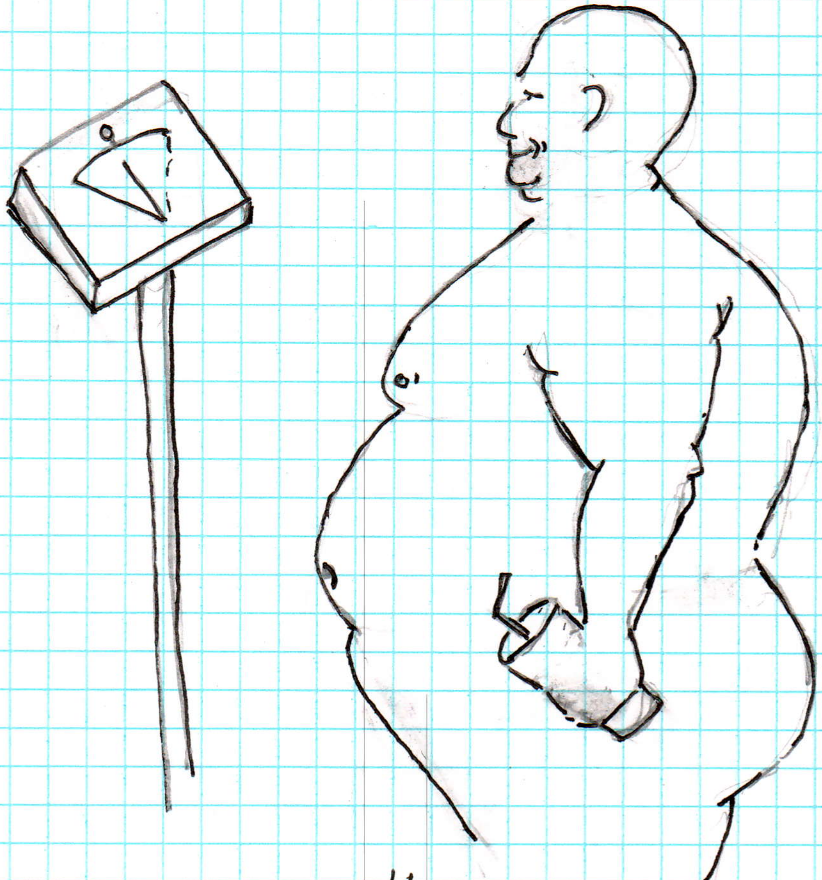
$(-a)(-b) = -(a(-b)) = ab > 0$

<u>Flaws of rational number system</u>

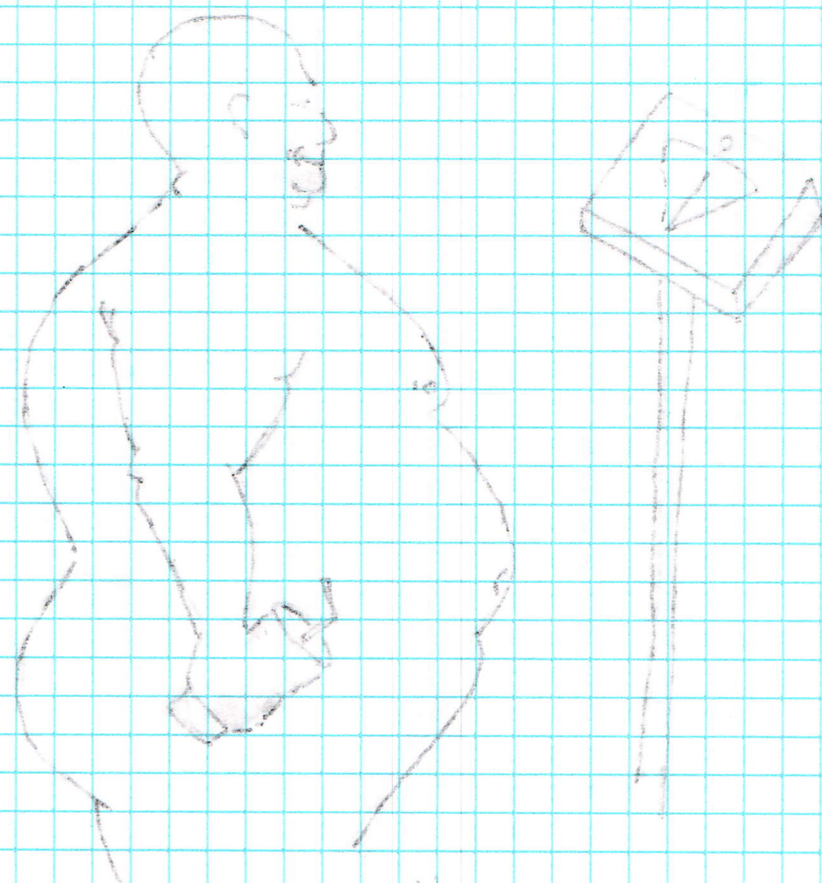
1) Unable to relate distinct parts of space to one another. (i.e. the length of a string cannot always be communicated in terms of the chosen unit length)

2) Unable to express cause and effect even in basic physics equations

3) Under closer scrutiny (measure theory) concepts like volume and mass, velocity and distance completely break down.



Using measure theory, you can prove that his weight is 0 lb. The proof is valid under the assumption that the "fatman" lives in $\mathbb{Q}^3$ - 3D rational number space.



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