HW. # 6

Homework problems are taken from "Principles of Mathematical Analysis" by W. Rudin and "Real Analysis" by N. L. Carothers. The problems are color coded to indicate level of difficulty. The color green indicates an elementary problem, which you should be able to solve effortlessly. Yellow means that the problem is somewhat harder. Red indicates that the problem is hard. You should attempt the hard problems especially.

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. Let $f:[0,\ 1]\to R$ be defined by:

   (a) $f(x)=\begin{cases} 0 & if \quad x\notin Q \\ x & if \quad x\in Q \end{cases}$

   (b) $f(x)=\begin{cases} 1-x & if \quad x\notin Q \\ x & if \quad x\in Q \end{cases}$

   (c) $f(x)=\begin{cases} 0 & if \quad x\notin Q \\ 1/n & if \quad x=\dfrac{m}{n}\in Q \quad (in \quad lowest \quad terms) \end{cases}$

   Determine the points at which f is continuous.

2. Given a subset A of some "universal" set S, we define $\chi_A : S\to R$, the **characteristic function** of A, by $\chi_A(x)=1$ if $x\in A$ and $\chi_A(x)=0$ if $x\notin A$. Prove or disprove the following formulas: $\chi_{A\cup B}=\chi_A+\chi_B$, $\chi_{A\cap B}=\chi_A\cdot\chi_B$, $\chi_{A\setminus B}=\chi_A-\chi_B$. What corrections are necessary?

3. If $f:A\to B$ and $C\subset B$, what is $\chi_C\circ f$ (as a characteristic function)?

4. Show that $\chi_\Delta : R\to R$, the characteristic function of the Cantor set, is discontinuous at each point of Δ.

5. If $A\subset R$, show that χ_A is continuous at each point of int(A). Are there any other points of continuity?

6. Let $f:R\to R$ be continuous. Show that $\{x: f(x)>0\}$ is an open subset of R and that $\{x: f(x)=0\}$ is a closed subset of R. If *f(x)* = 0 whenever x is rational, show that *f(x)* = 0 for every real x.

7. (a) If $f: M \to R$ is continuous and $a \in R$, show that the sets $\{x: f(x) > a\}$ and $\{x: f(x) < a\}$ are open subsets of M

(a) Conversely, if the sets $\{x: f(x) > a\}$ and $\{x: f(x) < a\}$ are open for every $a \in R$, show that f is continuous.

(b) Show that f is continuous even if we assume only that the sets $\{x: f(x) > a\}$ and $\{x: f(x) < a\}$ are open for every rational a.

8. Let $f: R \to R$ be continuous.

(a) If $f(0) > 0$, show that $f(x) > 0$ for all x in some interval (-a, a)

(b) If $f(x) \geq 0$ for every rational x, show that $f(x) \geq 0$ for all real x.

9. Let $A = (0,\ 1] \cup \{2\}$ be considered as a subset of R. Show that every function $f: A \to R$ is continuous, relative to *A*, at 2.

10. Let *A* and *B* be subsets of *M*, and let $f: M \to R$. Prove or disprove the following statements:

(a) If *f* is continuous at each point of *A* and *f* is continuous at each point of *B*, then *f* is continuous at each point of $A \cup B$.

(b) If $f|_A$ is continuous, relative to *A* and $f|_B$ is continuous, relative to *B*, then $f|_{A \cup B}$ is continuous, relative to $A \cup B$.

If either statement is not true in general, what modifications are necessary to make it so?

11. Let $I = (R \setminus Q) \cap [0,\ 1]$ with its usual metric. Prove that there is a continuous function *g* mapping *I* onto $Q \cap [0,\ 1]$.

12. Let $f, g: (M, d) \to (N, p)$ be continuous, and let *D* be a dense subset of *M*. If $f(x) = g(x)$ for all $x \in D$, show that $f(x) = g(x)$ for all $x \in M$. If *f* is onto, show that *f(D)* is dense in *N*.

13. A function $f: R \to R$ is said to satisfy a **Lipschitz condition** if there is a constant $K < \infty$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in R$. More economically, we may say that *f* is Lipschitz (or Lipschitz with constant K if a particular constant seems to matter). Show that sin x is Lipschitz with constant K = 1. Prove that a Lipschitz function is continuous.

14. A function $f: (M, d) \to (N, p)$ is called **Lipschitz** if there is a constant $K < \infty$ such that $p(f(x),\ f(y)) \leq Kd(x,\ y)$ for all $x, y \in M$. Prove that a Lipschitz mapping is continuous.

15. Show that the map $L(f) = \int_a^b f(t)dt$ is Lipschitz with constant K = b – a for $f \in C[a,\ b]$.

16. Define $g : \ell_2 \to R$ by $g(x) = \sum_{n=1}^{\infty} x_n / n$. Is g continuous?