HW. # 5

Homework problems are taken from "Principles of Mathematical Analysis" by W. Rudin and "Real Analysis" by N. L. Carothers. The problems are color coded to indicate level of difficulty. The color green indicates an elementary problem, which you should be able to solve effortlessly. Yellow means that the problem is somewhat harder. Red indicates that the problem is hard. You should attempt the hard problems especially.

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

2. Calculate $\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$.

3. If $s_1=\sqrt{2}$, and $s_{n+1}=\sqrt{2+\sqrt{s_n}}$ (n = 1, 2, 3, …), prove that $\{s_n\}$ converges and that $s_n<2$ for n = 1, 2, 3, …

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by $s_1=0$; $s_{2m}=\frac{s_{2m-1}}{2}$; $s_{2m+1}=\frac{1}{2}+s_{2m}$.

5. For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that $\limsup_{n\to\infty}(a_n+b_n)\le\limsup_{n\to\infty}(a_n)+\limsup_{n\to\infty}(b_n)$ provided the sum on the right is not of the form $\infty-\infty$.

6. Investigate the behavior (convergence or divergence) of $\sum a_n$ if

   (a) $a_n=\sqrt{n+1}-\sqrt{n}$;

   (b) $a_n=\dfrac{\sqrt{n+1}-\sqrt{n}}{n}$;

   (c) $a_n=\left(\sqrt[n]{n}-1\right)^n$;

   (d) $a_n=\dfrac{1}{1+z^n}$, for complex values of z.

7. Suppose $a_n\ge 0$. Prove that the convergence of $\sum a_n$ implies the convergence of $\sum\frac{\sqrt{a_n}}{n}$.

8. If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

9. Find the radius of convergence of each of the following power series:

(a) $\sum n^3 z^n$;

(b) $\sum \frac{2^n}{n!} z^n$;

(c) $\sum \frac{2^n}{n^2} z^n$;

(d) $\sum \frac{n^3}{3^n} z^n$;

10. Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

11. Suppose $a_n > 0$, $s_n = a_1 + \ldots + a_n$, and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

(b) Prove that $\frac{a_{N+1}}{s_{N+1}} + \ldots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$ and deduce that $\sum \frac{a_n}{s_n}$ diverges.

(c) Prove that $\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$ and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

(d) What can be said about $\sum \frac{a_n}{1+na_n}$ and $\sum \frac{a_n}{1+n^2 a_n}$?

12. Suppose $a_n > 0$ and $\sum a_n$ converges. Put $r_n = \sum_{k=n}^{\infty} a_k$.

(a) Prove that $\frac{a_m}{r_m} + \ldots + \frac{a_n}{r_n} \geq 1 - \frac{r_n}{r_m}$ if m < n, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

(b) Prove that $\frac{a_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)$ and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

13. If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \ldots + s_n}{n+1} \quad \text{(n = 0, 1, 2, …)}$$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

(b) Construct a sequence $\{s_n\}$, which does not converge, although $\lim \sigma_n = 0$.

(c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?

(d) Put $a_n = s_n - s_{n-1}$, for $n \ge 1$. Show that $s_n - \sigma_n = \frac{1}{n+1}\sum_{k=1}^{n} ka_k$. Assume that $\lim(na_n) = 0$ and that σ_n converges. Prove that $\{s_n\}$ converges. [This gives a converse of (a), but under the additional assumption that $na_n \to 0$.]

(e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|na_n| \le M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$, by completing the following outline:

If m < n, then $s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{k=m+1}^{n}(s_n - s_k)$. For these k,

$$|s_n - s_k| \le \frac{(n-k)M}{k+1} \le \frac{(n-m-1)M}{m+2}.$$

Fix $\varepsilon > 0$ and associate with each n the integer m that satisfies

$$m \le \frac{n-\varepsilon}{1+\varepsilon} < m+1.$$

Then $(m+1)/(n-m) \le 1/\varepsilon$ and $|s_n - s_k| < M\varepsilon$. Hence $\limsup_{n\to\infty}|s_n - \sigma| \le M\varepsilon$. Since ε was arbitrary, $\lim s_n = \sigma$.