HW. # 2

Homework problems are taken from “Principles of Mathematical Analysis” by W. Rudin and “Real Analysis” by N. L. Carothers. The problems are color coded to indicate level of difficulty. The color green indicates an elementary problem, which you should be able to solve effortlessly. Yellow means that the problem is somewhat harder. Red indicates that the problem is hard. You should attempt the hard problems especially.

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. Check that the relation “is equivalent to” defines an equivalence relation. That is, show that (i) A ~ A, (ii) A ~ B if and only if B ~ A, and (iii) if A ~ B and B ~ C, then A ~ C.

2. Given finitely many countable sets $A_1,\ldots,A_n$, show that
   (a) $A_1 \cup \ldots \cup A_n$ is countable.
   (b) $A_1 \times \ldots \times A_n$ is countable.

3. Prove that a set is infinite if and only if it is equivalent to a proper subset of itself. [Hint: If A is infinite and $x \in A$, show that A is equivalent to A\{x}.]

4. If A is infinite and B is countable, show that A and $A \cup B$ are equivalent. [Hint: No containment relation between A and B is assumed here.]

5. Let A be countable. If $f : A \to B$ is onto, show that B is countable; if $g : C \to A$ is one-to-one, show that C is countable. [Hint: Be careful!]

6. Show that (0, 1) is equivalent to [0, 1] and to R.

7. Show that (0, 1) is equivalent to the unit square (0, 1)×(0, 1). [Hint: “Interlace” decimals—but carefully!]

8. Prove that (0, 1) can be put into one-to-one correspondence with the set of all *functions* $f : N \to \{0,\ 1\}$.

9. Show that the set of all functions $f : A \to \{0,\ 1\}$ is equivalent to $P(A)$, the power set of A (i.e. the set of all subsets of A).

10. Show that N contains infinitely many pairwise disjoint infinite subsets.

11. Show that any collection of pairwise disjoint, nonempty open intervals in R is at most countable. [Hint: Each one contains a rational!]

12. Prove that N contains uncountably many infinite subsets $\{N_\alpha\}_{\alpha \in R}$ such that $N_\alpha \cap N_\beta$ is finite if $\alpha \neq \beta$ (This problem appeared on the Putnam Mathematical Competition. It is considered very hard).