# HW. # 1

Homework problems are taken from "Principles of Mathematical Analysis" by W. Rudin and "Real Analysis" by N. L. Carothers. The problems are color coded to indicate level of difficulty. The color green indicates an elementary problem, which you should be able to solve effortlessly. Yellow means that the problem is somewhat harder. Red indicates that the problem is hard. You should attempt the hard problems especially.

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. If r is a rational ($r \neq 0$) and x is irrational, prove that r + x and rx are irrational.

2. Prove that there is no rational number whose square is 12.

3. Let E be a nonempty subset of an ordered set; suppose $\alpha$ is a lower bound and $\beta$ is an upper bound of E. Prove that $\alpha \leq \beta$.

4. Let A be a nonempty set of real numbers which is bounded below. Let –A be the set of all numbers –x, where $x \in A$. Prove that inf A = -sup(-A) .

5. Fix b>1.
   (a) If m, n, p, q are integers, n > 0, q > 0, and r = m/n = p/q, prove that
   $$(b^m)^{1/n} = (b^p)^{1/q}$$
   Hence it makes sense to define $b^r = (b^m)^{1/n}$.
   (b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
   (c) If x is real, define B(x) to be the set of all numbers $b^t$, where t is rational and $t \leq x$. Prove that
   $$b^r = \sup B(r)$$
   when r is rational. Hence it makes sense to define
   $$b^x = \sup B(x)$$
   for every real x.
   (d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

6. Fix b >1 , y > 0, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the logarithm of y to the base b.)
   (a) For any positive integer n, $b^n - 1 \geq n(b-1)$.
   (b) Hence $b - 1 \geq n(b^{1/n} - 1)$.
   (c) If t > 1 and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

(d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n; to see this, apply part (c) with $t = yb^{-w}$.

(e) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

(f) Let A be the set of all w such that $b^w < y$, and show that x = sup A satisfies $b^x = y$.

(g) Prove that this x is unique.

7. Let $p \geq 2$ be a fixed integer, and let 0 < x < 1. If x has a finite-length base p decimal expansion, that is, if $x = a_1 / p + ... + a_n / p^n$ with $a_n \neq 0$, prove that x has precisely *two* base p decimal expansions. Otherwise, show that the base p decimal expansion for x is unique.

8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.

9. Suppose $z = a + bi$, $w = c + di$. Define z < w if a < c, and also if a = c but b < d. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?

10. Suppose $z = a + bi$, $w = u + vi$, and

$$a = \left( \frac{|w| + u}{2} \right)^{1/2}, \ b = \left( \frac{|w| - u}{2} \right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

11. If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that z = rw. Are w and r always uniquely determined by z?

12. If $z_1, ..., z_n$ are complex, prove that

$$|z_1 + z_2 + ... + z_n| \leq |z_1| + |z_2| + ... + |z_n|.$$

13. If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2.$$