NAME:

Summer 2019 Math 351 Exam 2

Instructions: WRITE YOUR NAME CLEARLY. Do as many problems as you can for a maximal score of 100. SHOW YOUR WORK!

1. True, False, or incoherent

   a) All complex-valued sequences with a finite range are convergent sequences. [2 pts]

   b) Let $\{s_n\}$ and $\{t_n\}$ be complex-valued sequences such that $\lim_{n\to\infty}(s_n + t_n)$ = L. Then $\{s_n\}$ and $\{t_n\}$ must be convergent sequences.

   [2 pts]

   c) For any sequence of real numbers $\{a_n\}$, the inequality $\liminf a_n \le \limsup a_n$ always holds. [2 pts]

   d) There exists a sequence of real numbers $\{a_n\}$, for which $T_n = \sup\{a_k : k \ge n\}$ is a strictly increasing sequence. [2 pts]

   e) Given a sequence $\{a_n\}$ in an arbitrary metric space (M, d), we can always compute limit supremum and limit infimum. [2 pts]

2. True, False, or incoherent

   a) Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms. If the root test gives no information, then it is useless to try the ratio test. [2 pts]

   b) Suppose that $\sum_{n=1}^{\infty} a_n$ is a real-valued series such that for every $\varepsilon > 0$, there is an integer N, for which $\sum_{n=N}^{\infty} a_n < \varepsilon$. Then we may conclude that the series converges. [2 pts]

   c) Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms. If the ratio test gives no information, then it is useless to try the root test. [2 pts]

d) A series of non-negative real numbers $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges. [2 pts]

e) The series $\frac{1}{2}+1+\frac{1}{8}+\frac{1}{4}+\frac{1}{32}+\frac{1}{16}+\ldots+$ converges to 2. [2 pts]

3. True, False, or incoherent

a) The Cantor function is decreasing. [2 pts]

b) A discrete metric space does not have any nowhere dense subsets. [2 pts]

c) Every dense set is perfect. [2 pts]

d) The compliment of a nowhere dense set is dense. [2 pts]

e) In any metric space, every nonempty perfect set is infinite. [2 pts]

4. True, False, or incoherent

a) Let $\{x_n\}$ be a sequence of real numbers and $\{\varepsilon_n\}$ be a corresponding sequence of positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Then the function f defined by $f(x) = \sum_{x_n \geq x} \varepsilon_n$ is increasing. [2 pts]

b) The function $f(x) = x^2$ is continuous. [Hint: be careful!] [2 pts]

c) If $f(x_n) \to f(x)$ for every continuous function *f*: (M, d) →**R**, then it must be the case that $x_n \to x$. [2 pts]

d) If *f*: (M, d) → (N, p) is invertible with a continuous inverse f^{-1}, then for any open subset of M, V, *f* (V) must be an open subset of N. [2 pts]

e) Let $X_\Delta : \mathbf{R} \to \mathbf{R}$ be the characteristic function of the Cantor set. Then X_Δ is discontinuous at every point of the Cantor set. [2 pts]

5. True, False, or incoherent

a) Let *f*: (M, d) → (N, p) be a function and suppose that V is a subset of N that contains a neighborhood of *f* (x). If $[f^{-1}(V)]^\circ = \emptyset$, then *f* is **not** continuous at x. [2 pts]

b) Let M be a discrete metric space. Then any function *f* : M → **R** is continuous. [2 pts]

c) Let d and p be equivalent metrics. Then the set of real-valued continuous functions on (M, d) is equivalent to the set of real-valued continuous functions on (M, p). [2 pts]

d) For any metric space (M, d), there exists some function *f*: (M, d) → **R** such that for any real number *a*, the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are open, but the function is **not** continuous. [2 pts]

e) Suppose that M = A ∪ B, where A ∩ B = ∅. If *f*: (A, d) → **R** and *f*: (B, d) → **R** are continuous, then *f*: (M, d) → **R** must be continuous. [2 pts]

6. True, False, or incoherent

a) The empty set ∅ is connected. [2 pts]

b) Let ϑ be a collection of connected sets. Then $\cap\vartheta$ is necessarily connected. [2 pts]

c) If A is connected in (M, d), then $\overline{A}$ is connected in (M, d). [2 pts]

d) If $\overline{A}$ is connected in (M, d), then A is connected in (M, d). [2 pts]

e) Suppose that for any continuous function $f : M \to \mathbf{R}$, $f(M)$ is a connected subset of **R**. Then M is necessarily connected. [2 pts]

7. Construct a real-valued sequence $\{a_n\}$ such that $\limsup a_n = 5$, while $\liminf a_n = -1$. Can such a sequence converge? [10 pts]

8. Let $f : [0, 1] \to R$ be defined by $f(x) = \begin{cases} 2x - 1 & if \quad x \notin Q \\ x^2 & if \quad x \in Q \end{cases}$. Determine the points at which f is continuous. [10 pts]

9. Prove that the set $\{(x,\ y): x^3 \geq y^5\}$ is closed as a subset of $\mathbf{R}^2$.

[10 pts]

10. Suppose that *a* is an isolated point of (M, d). Prove or disprove: There exists a function *f*: (M, d) → **R** that is **not** continuous at the point *a*.

[10 pts]

11. Let θ be an irrational number and define $f: \mathbb{R} \to [0, 1)$ by $f(x) = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer function. (In other words $f(x)$ is the fractional part of x. For example $f(3.1415 \ldots) = 0.1415 \ldots$). Show that $D = \{f(n\,\theta): n \in \mathbb{N}\}$ is a dense subset of $[0, 1]$. [10 pts]

12. Let $\{x_n\}$ be a sequence of real numbers defined by

$$x_n = \begin{cases} \dfrac{1}{2k} & if \quad n = 2k-1 \\ \dfrac{1}{2k-1} & if \quad n = 2k \end{cases}$$ where $k \geq 1$ and therefore $n \geq 1$. Set

$\varepsilon_n = \left(\dfrac{1}{2}\right)^n$ and define $f : R \to R$ by

$$f(x) = \sum_{n\,:\,x_n < x} \varepsilon_n$$

(a) Compute $f(0)$, $f(-1)$, $f(1)$, $f(\sqrt{2})$, and $f(1/2)$. [4 pts]

(b) Determine the set of discontinuities, D(f), for the function. Justify your claim. [6 pts]