NAME:

Summer 2019 Math 351 Exam 1

Instructions: WRITE YOUR NAME CLEARLY. Do as many problems as you can for a maximal score of 100. SHOW YOUR WORK!

1. True, False, or incoherent
   a) If A is a bounded subset of the real numbers, then sup(A) is a real number. [2 pts]

   b) The field of complex numbers can be made into an ordered set. [2 pts]

   c) Every ordered set that has the least upper bound property also has the greatest lower bound property. [2 pts]

   d) Suppose that x = 0.101001000100001… is an infinite expansion in base 10. Then x has another representation in base 10. [2 pts]

   e) No number in (0, 1) has more than two decimal expansions in base p. [2 pts]

2. True, False, or incoherent
   a) A finite Cartesian product of countable sets is always countable. [2 pts]

   b) A countable Cartesian product of countable sets is countable. [2 pts]

   c) The cardinality of the power set $P(\mathbb{R})$ is bigger than card(A), where A is the set of all functions $f\colon \mathbb{R} \to \{0, 1\}$ [2 pts]

   d) Every infinite set has a proper subset of the same cardinality. [2 pts]

e) Every irrational number is a root of some polynomial with integer coefficients. [2 pts]

3. True, False, or incoherent

a) If d and p are metric functions on M, then so is $\sigma = \sqrt{d + p}$.
[2 pts]

b) If ℝ is equipped with the discrete metric, then diam (0, 4) = 4.
[2 pts]

c) {1/n} is a Cauchy sequence. [2 pts]

d) Every Cauchy sequence is convergent. [2 pts]

e) Equivalent metrics preserve Cauchy sequences. That is, if d and p are equivalent on M and $\{x_n\}$ is a sequence in M, then $\{x_n\}$ is Cauchy under the metric d if and only if $\{x_n\}$ is Cauchy under the metric p.
[2 pts]

4. True, False, or incoherent

a) An infinite intersection of open sets is never open. [2 pts]

b) All sets are either open or closed. [2 pts]

c) If ℝ is equipped with the discrete metric, the set (0, 1) is closed.
[2 pts]

d) Let F be a subset of ℝ, then the set of all limit points of F, $F'_{regular}$, under the regular metric is the same as F'_d — the set of all limit points of F under the metric $d(x,\ y) = \dfrac{|x-y|}{1+|x-y|}$. [2 pts]

e) No convergent sequence has more than one limit point.
[2 pts]

5. True, False, or incoherent

a) An open set cannot have any limit points. [2 pts]

b) A closed set must contain the limits of all of its sequences. [2 pts]

c) If cl(A) = A, then A must be closed. [2 pts]

d) In an arbitrary metric space, any nonempty open set can be written as an (at most) countable union of disjoint open balls. [2 pts]

e) There exist metric spaces in which finite sets are not closed.
[2 pts]

6. Which of the following are metric functions on $(0, \infty)$? Write simply metric or not metric.

a) $d(x,\ y) = \left|\frac{1}{x^4} - \frac{1}{y^4}\right|$ [2 pts]

b) $d(x,\ y) = |x - 3y|$ [2 pts]

c) $d(x,\ y) = \sqrt{|x-y|} + \frac{|x-y|}{1+|x-y|}$ [2 pts]

d) $d(x,\ y) = \tan^{-1}|x-y|$ [2 pts]

e) $d(x,\ y) = \min\left\{|x-y|^{3/4},\ 2\right\}$ [2 pts]

7. Let $0 < \alpha < 1$. Show that if x and y are positive real numbers, then $\left|x^{\alpha} - y^{\alpha}\right| \leq |x - y|^{\alpha}$. [Hint: $d(x,\ y) = |x - y|^{\alpha}$ defines a metric on $\mathbb{R}$]

[10 pts]

8. Let M = (0, ∞) be supplied with the metric function $d(x,\ y) = \left|\tan^{-1} x - \tan^{-1} y\right|$ and let $\{n\}_{n=1}^{\infty}$ be a sequence of positive integers.

   a) Is the sequence $\{n\}_{n=1}^{\infty}$ a Cauchy sequence in (M, d)? Justify your answer.

   [6 pts]

   b) Does the sequence $\{n\}_{n=1}^{\infty}$ converge in (M, d)? [4 pts]

9. Let (M, d) be a metric space. Prove that an open ball of (M, d) is always an open set of (M, d) [10 pts]

10. Decide whether the set $\bigcup_{n=1}^{\infty}[4n,\ 4n+1]$ is closed, open, or neither as a subset of $\mathbb{R}$. Justify your answer. [10 pts]

11. Prove Cantor's theorem. In other words, show that for any set A, the power set P(A) always has larger cardinality than A. [10 pts]

12. Let $p \geq 2$ be a fixed integer, and let $0 < x < 1$. If x has a finite-length base p decimal expansion, that is, if $x = a_1 / p + ... + a_n / p^n$ with $a_n \neq 0$, prove that x has precisely *two* base p decimal expansions. Otherwise, show that the base p decimal expansion for x is unique. [10 pts]

13. Generalize Young's inequality. In other words, show that for any vector $(p_1, p_2, \ldots, p_n)$ of positive numbers satisfying $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$ and positive real numbers $a_1, a_2, \ldots a_n$, one has $a_1 a_2 \ldots a_n \leq \frac{{a_1}^{p_1}}{p_1} + \frac{{a_2}^{p_2}}{p_2} + \cdots + \frac{{a_n}^{p_n}}{p_n}$. [10 pts]