Review For Exam 2

Instructions: The exam attempts to measure your level of understanding from basic to advanced and is therefore divided into 2 types of problems. 50% of the exam will consist of a subset of the true/false problems listed below. The other 50% will be made up from the homework problems, assignments, and other questions mentioned on the review list. This will, hopefully, make it hard to fail and hard to get a perfect grade.

Chapter 2

Lecture Notes to carefully study

- **Nested Interval** Chapter 2 (part d) 73-76
- **Perfect Sets** Chapter 2 (part d) 76-82
- **Cantor Set** Chapter 2 (part d) 83-89

Homework Problems

- HW 4 questions 12-16.

Hand-In Assignment Problems

- Study problems 1-4 on Hand-In Assignment 4

Comprehension Problems for Chapter 2

1. Determine which of the following intersections of subsets of **R** must produce a nonempty set. Which of these operations result in a set that contains only one point?

   (a) S = $\bigcap_{n=1}^{\infty}[n,\ 1+n^3]$

   (b) S = $\bigcap_{n=1}^{\infty}[n/(n+1),\ 2)$

   (c) S = $\bigcap_{n=1}^{\infty}[a,\ b_n]$; $b_0 = b$, $b_n = \dfrac{a+b_{n-1}}{2}$.

2. Let E be the set of all x ∈ [0, 1] whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in [0, 1]? Is E perfect?
3. Let Δ_α be a generalized Cantor set, in which the middle open interval of length $\alpha/3$ is removed during the first step, two middle open intervals of length $\alpha/3^2$ are removed in the second step, and, in general, 2^{n-1} middle intervals of length $\alpha/3^n$ are removed in the nth step. What is the length of Δ_α?

True, False or Incoherent?

1. The nested interval theorem guarantees that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ for any sequence of nested intervals $I_1 \supset I_2 \supset I_3 \supset ... \supset I_n \supset ....$ of **R**.
2. In any metric space, every nonempty perfect set is infinite.
3. In any metric space, every nonempty perfect set is uncountable.
4. Every dense set is perfect.
5. If P is a perfect subset of (M, d), then cl(P) = M.
6. **R** has a nonempty perfect subset that contains no rational numbers.
7. A nowhere dense set cannot have any limit points.
8. The compliment of a nowhere dense set is dense.
9. Suppose that A is a subset of some metric space (M, d) and that int(A) = ∅. Then A^c is a dense subset of M.
10. Suppose that A is a subset of some metric space (M, d) and that int(A) = ∅. Then A is nowhere dense in M.
11. A discrete metric space has no proper dense subsets.
12. A discrete metric space does not have any nowhere dense subsets.
13. The compliment of a dense set is nowhere dense.
14. The number whose decimal expansion is 0.0021 in base 3 is not an element of the Cantor set.
15. The number whose decimal expansion is 2.022022022… in base 3 is an element of the Cantor set.
16. The Cantor set is open in **R**.
17. The Cantor set is perfect in **R**.
18. The Cantor set is nowhere dense in **R**.
19. The Cantor function is decreasing.

Chapter 3

Lecture Notes to carefully study

▪ **Basic Limits**	Chapter 3	1-5.
▪ **Upper and Lower Limits**	Chapter 3	5-7.
▪ **Special Sequences**	Chapter 3	7-9
▪ **Series**	Chapter 3	9-34

Homework Problems

- HW 5 questions 1-9

Hand-In Assignment Problems

- Study problems 1-4 on Hand-In Assignment 5

Comprehension Problems for Chapter 3

1. Suppose the real-valued sequence $\{a_n\}$ is increasing. What is the relationship between $\lim_{n\to\infty} a_n$ and $\limsup_{n\to\infty}(a_n)$?

2. Suppose that the real-valued sequence $\{a_n\}$ is given by the formula

$$a_n = \begin{cases} \dfrac{2k}{k+1} & if \quad n = 4k \\ (-1)^k & if \quad n = 4k+1 \\ \displaystyle\sum_{j=1}^{k} \frac{1}{j!} & if \quad n = 4k+2 \\ 3^{-k} & if \quad n = 4k+3 \end{cases}$$

   Compute $\liminf_{n\to\infty}(a_n)$ and $\limsup_{n\to\infty}(a_n)$.
3. Suppose that a real-valued sequence $\{x_n\}$ is convergent with $x_n \to x$. What is $\liminf_{n\to\infty}(x_n)$ and $\limsup_{n\to\infty}(x_n)$? Justify your answer.
4. Let $L_1 < L_2 < ... < L_n$ and suppose that any subsequence $\{x_{n(k)}\}_{k=1}^{\infty}$ of the real-valued sequence $\{x_n\}$ has a further subsequence $\{x_{n(k(t))}\}_{t=1}^{\infty}$ that converges to one of the L_j. Find $\liminf_{n\to\infty}(x_n)$ and $\limsup_{n\to\infty}(x_n)$. Justify your answer.

True, False or Incoherent?

1. All complex-valued sequences with a finite range are convergent sequences.
2. Let $\{s_n\}$ and $\{t_n\}$ be complex-valued sequences such that $\lim_{n\to\infty}(s_n + t_n) =$ L. Then $\{s_n\}$ and $\{t_n\}$ must be convergent sequences.
3. Every increasing sequence of real numbers is convergent.
4. Given a sequence $\{a_n\}$ in an arbitrary metric space (M, d), we can always compute limit supremum and limit infimum.
5. For any sequence of real numbers $\{a_n\}$, the inequality $\liminf a_n \le \limsup a_n$ always holds.
6. There exists a sequence of real numbers $\{a_n\}$, for which $T_n = \sup\{a_k : k \ge n\}$ is a strictly increasing sequence.

7. Suppose that $\sum_{n=1}^{\infty} a_n$ is a real-valued series such that for every $\varepsilon > 0$, there is an integer N, for which $\sum_{n=N}^{\infty} a_n < \varepsilon$. Then we may conclude that the series converges.
8. A series of non-negative real numbers $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.
9. Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms. If $\limsup \frac{a_{n+1}}{a_n} > 1$, the series diverges.
10. Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms. If the ratio test gives no information, then it is useless to try the root test.
11. Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms. If the root test gives no information, then it is useless to try the ratio test.
12. The series $\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + ... +$ converges to 2.

Chapter 4

Lecture Notes to carefully study

▪ **Limits and Continuity**	Chapter 4 (part a) 1-6
▪ **Abstract Continuity**	Chapter 4 (part a) 6-10
	Chapter 4 (part b) 11-12
▪ **Connected Sets**	Chapter 4 (part b) 12-19

Homework Problems

- HW 6 questions 1-15

Comprehension Problems for Chapter 4

1. Let f: A → B be any function and let $A_0 \subset A$ and $B_0 \subset B$.
   (a) Show that $A_0 \subset f^{-1}(f(A))$ and that equality holds if f is injective.
   (b) Show that $f(f^{-1}(B_0)) \subset B_0$ and that equality holds if f is surjective.

2. Let f : A → B be any function and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that f^{-1} preserves inclusions, unions, intersections, and differences of sets:
   (a) $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$.
   (b) $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$.
   (c) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.
   (d) $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$.
   Show that f preserves inclusions and unions only:
   (e) $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$.
   (f) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$.
   (g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$; show that equality holds if f is injective.
   (h) $f(A_0 - A_1) \supset f(A_0) - f(A_1)$; show that equality holds if f is injective.
3. Show that (b), (c), (f), and (g) of exercise 2 hold for arbitrary unions and intersections.
4. Let f : (M, d) → (N, p) be a function. Let x ∈ M. Decide whether the information below is sufficient to conclude that f is continuous or that f is discontinuous at x. Justify your answer.
   (a) $f^{-1}(V)$ is open whenever V is an open neighborhood of f (x).
   (b) $f^{-1}(V)$ is closed whenever V is an open neighborhood of f (x).
   (c) $f^{-1}(V)$ contains an open neighborhood of x whenever V is an open neighborhood of f (x).
   (d) $[f^{-1}(V)]^\circ = \emptyset$ for some open neighborhood V of f (x).
   (e) $[f^{-1}(V)]^\circ \neq \emptyset$ for all open neighborhoods V of f (x).
   (f) $f(x_n)$ is a convergent sequence whenever $x_n \xrightarrow{d} x$.
   (e) f maps every Cauchy sequence $x_n \in$ (M, d) to a Cauchy sequence $f(x_n) \in$ (N, p).
   (f) $x_n \in$ (M, d) is a Cauchy sequence, whenever $f(x_n) \in$ (N, p) is Cuauchy.

<u>Hand-In Assignment Problems</u>

- Study problems 1-4 on Hand-In Assignment 6

<u>True, False or Incoherent?</u>

1. All increasing functions of the form f: (a, b) → **R** must have jump discontinuities.
2. A function f: (a, b) → **R** that has only jump discontinuities has at most countably many points of discontinuity. [This is very hard!]

3. If f: (a, b) → **R** is discontinuous at some point x = c. This discontinuity is necessarily a jump discontinuity.
4. Let $\{x_n\}$ be a sequence of real numbers and $\{\varepsilon_n\}$ be a corresponding sequence of positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Then the function f defined by $f(x) = \sum_{x_n \geq x} \varepsilon_n$ is increasing.
5. The function $f(x) = x^2$ is continuous. [Hint: be careful!]
6. Let f: (M, d) → (N, p) be continuous. Then $x_n \to x$ whenever $f(x_n) \to f(x)$.
7. If $f(x_n) \to f(x)$ for every continuous function f: (M, d) → **R**, then it must be the case that $x_n \to x$.
8. Let f: (M, d) → (N, p) be continuous. If E is a closed subset of M, then it must be the case that f (E) is closed in N.
9. If f: (M, d) → (N, p) is invertible with a continuous inverse f^{-1}, then for any open subset of M, V, f (V) must be an open subset of N.
10. Let X_Δ: **R** → **R** be the characteristic function of the Cantor set. Then X_Δ is discontinuous at every point of the Cantor set.
11. Let f: (M, d) → (N, p) be a function and suppose that V is a subset of N that contains a neighborhood of f (x). If $[f^{-1}(V)]^\circ = \emptyset$, then f is **not** continuous at x.
12. Let M be a discrete metric space. Then any function f : M → **R** is continuous.
13. Let d and p be equivalent metrics. Then the set of real-valued continuous functions on (M, d) is equivalent to the set of real-valued continuous functions on (M, p).
14. For any continuous real-valued function f on (M, d), the set $\{x : f(x) \geq 0\}$ is open in M.
15. For any metric space (M, d), there exists some function f: (M, d) → **R** such that for any real number a, the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are open, but the function is **not** continuous.
16. If f: (M, d) → **R** is **not** continuous. Then f: (A, d) → **R** is not continuous for any subset A of M.
17. Suppose that M = A ∪ B, where A ∩ B = ∅. If f: (A, d) → **R** and f: (B, d) → **R** are continuous, then f: (M, d) → **R** must be continuous.
18. Suppose that x is an isolated point of (M, d). Then any function f: (M, d) → **R** must be continuous at x.
19. There exist Lipschitz functions that are not continuous.
20. The empty set ∅ is connected.
21. The Cantor set is connected.

22. Every connected subset of **R** that contains at least 2 elements is uncountable.
23. Let ϑ be a collection of connected sets. Then $\cap\vartheta$ is necessarily connected.
24. If A is connected in (M, d) and B is connected in (N, p), then A × B is connected in $M \times N$.
25. If A is connected in (M, d), then $\overline{A}$ is connected in (M, d).
26. If $\overline{A}$ is connected in (M, d), then A is connected in (M, d).
27. Suppose that for any continuous function $f : M \to \mathbf{R}$, $f(M)$ is a connected subset of **R**. Then M is necessarily connected.