Review For Exam 1

Instructions: The exam attempts to measure your level of understanding from basic to advanced and is therefore divided into 2 types of problems. 50% of the exam will consist of a subset of the true/false problems listed below. The other 50% will be made up from the homework problems, assignments, and other questions mentioned on the review list. This will, hopefully, make it hard to fail and hard to get a perfect grade.

Chapter 1

Lecture Notes to carefully study

- Chapter 1 pages 1-26.

Homework Problems

- HW 1 questions 1 - 4, 7 – 9.
- HW 1 questions 5 – 6 are VERY important for the development of the subject as well as for your mastery of analytic techniques, but they will not be featured on the exam.

Hand-In Assignment Problems

- Study problems 1-4 on Hand-In Assignment 1

Comprehension Problems for Chapter 1

1. Determine which of the following sets have the least upper bound property and which have the greatest lower bound property.
   (a) $S = (-\infty, 1) \cup [2, 3) \cup (3, 10]$
   (b) $S = (-\infty, 1) \cup [2, 3) \cup [3, 10]$
   (c) $S = (-\infty, 1) \cup [2, 3) \cup [9, 10]$
2. Let $A = \{x \in \mathbf{Q}\colon x^2 < 12\}$. Does A have an upper bound in **Q**? If so, does A have a least upper bound in **Q**?
3. Repeat exercise 2 when A is considered as a subset of **R**.
4. Let S be an ordered set with the **greatest lower bound property**. True or false: S has the **least upper bound property**. Justify your answer.
5. Let E be a subset of nonnegative numbers in **R**. For $n \geq 1$, define $E^n = \{x_1 \cdot x_2 \cdot \ldots \cdot x_n : x_k \in E \text{ for each } k\}$ and $E^{\otimes n} = \{x^n : x \in E\}$.
   (a) What can you say about $\sup E^n$ in terms of sup E and inf E?
   (b) What can you say about $\sup E^{\otimes n}$ in terms of sup E and inf E?
6. Which of the following operations makes sense in **Q**
   (a) $(25^{1/2})^3$

(b) $(25^{1/6})^9$

Justify your answer. Be sure to consult problems 5 and 6 in HW 1.

7. Use the algorithm described on pages 20-21 of the lecture notes on chapter 1 to compute the decimal expansion of ½ when
   (a) p = 2
   (b) p = 3
   (c) p = 10
8. The real number 1/5 can be written as 0.2 (mod 10) and as 0.199999…(mod 10). Are there any other representations of 1/5 (mod 10)? Justify your answer. [Hint: See exercise 7 in HW 1]
9. Can the set of complex numbers be ordered? Can it be made into an ordered field?

True, False or Incoherent?

1. If S is an ordered set and $A \subseteq S$, then sup(A) and inf(A) always make sense.
2. If A is a subset of the real numbers, then sup(A) always makes sense.
3. If A is a subset of the real numbers, then sup(A) is a real number.
4. If the empty set $\emptyset$ is considered as a subset of the real numbers, then $\sup(\emptyset) = \infty$.
5. The field of complex numbers can be made into an ordered *set*.
6. The field of complex numbers can be made into an ordered *field*.
7. Every ordered field has the least upper bound property.
8. Every ordered set that has the least upper bound property also has the greatest lower bound property.
9. Every ordered set that is bounded has a greatest lower bound.
10. Suppose that x = 0.101001000100001… is an infinite expansion in base. Then x has another representation in base 10.
11. Every rational number in (0, 1) has more than one decimal expansion in base 10.
12. No number in (0, 1) has more than two decimal expansions in base p.
13. Every number in (0, 1) that has a finite expansion in base p also has an infinite decimal expansion in base p.

Chapter 2

Lecture Notes to carefully study

- **Cardinality** Chapter 2 (part a) 1-24. (You will not be required to prove Bernstein's theorem)
- **Metric Spaces** Chapter 2 (part a) 24-32.

- **Normed Spaces** Chapter 2 (part b) 33-41 (You will not be tested on l_p spaces, but you should know how to prove the "name" inequalities for the space $\mathbf{R}^n$.
- **Limits** Chapter 2 (part b) 41-50
- **Subsequences** Chapter 2 (part c) 51-55
- **Eq. Metrics** Chapter 2 (part c) 55-59
- **Open Sets** Chapter 2 (part c) 59-65
- **Closed Sets** Chapter 2 (part c) 65-72
- **Nested Interval** Chapter 2 (part d) 73-76
- **Perfect Sets** Chapter 2 (part d) 76-82

Homework Problems

- HW 2 questions 1-9, 11
- Question 11 is very significant in the study of analysis. Question 12 was featured on the Putnam exam.
- HW 3 questions 1-2, 4, 5-7, 10-16
- HW 4 questions 1-11

Hand-In Assignment Problems

- Study problems 1-4 on Hand-In Assignment 2
- Study problems 1-4 on Hand-In Assignment 3 (Regular)

Comprehension Problems for Chapter 2

1. Determine, for each of the following sets, whether or not it is countable. Justify your answers.
   (a) The set A of all functions f: {0, 1} → **N**
   (b) The set B_n of all functions f: {1, …, n} → **N**
   (c) The set C = $\bigcup_{n \in N} B_n$.
   (d) The set D of all functions f: **N** → **N**.
   (e) The set E of all functions f: **N** → {0, 1}.
   (f) The set F of all functions f: **N** → {0, 1} that are "eventually zero." [We say that f is **eventually zero** if there is a positive integer K such that f (n) = 0 for all n ≥ K.]
   (g) The set G of all functions f: **N** → **N** that are eventually 1.
   (h) The set H of all functions f: **N** → **N** that are eventually constant.
   (i) The set I of all two-element subsets of **N**.
   (j) The set J of all finite subsets of **N**.

2. Prove that the power set P(A) is always larger than A.
3. Show that in any metric space (M, d), $|d(x,a) - d(y,a)| \leq d(x,y)$.

4. Use the above result to show that $\left|\sqrt{x}-\sqrt{y}\right| \le \sqrt{|x-y|}$. In fact, show that $\left|x^{\alpha}-y^{\alpha}\right| \le |x-y|^{\alpha}$ for any $0<\alpha<1$
5. Prove the Cauchy-Schwarz, Young, Hoelder, and Minkowski Inequalities for the vector space $\mathbf{R}^n$ **.**

<u>True, False or Incoherent?</u>

1. A <u>*countable union*</u> of countable sets is countable.
2. An <u>*arbitrary union*</u> of countable sets is countable.
3. A <u>*finite Cartesian product*</u> of countable sets is always countable.
4. A <u>*countable Cartesian product*</u> of countable sets is countable.
5. All <u>*countable infinities*</u> are of the same size.
6. All <u>*uncountable infinities*</u> are of the same size.
7. The cardinality of the power set $P(\mathbf{R})$ is bigger than card(A), where A is the set of all functions f: $\mathbf{R} \to \{0, 1\}$.
8. card(**R**) < card(C[0, 1]). [This is hard!]
9. Every infinite set has a proper subset of the same cardinality.
10. Cantor's diagonalization argument may be used to show that one uncountably infinite set is bigger than another uncountably infinite set.
11. Every irrational number is a root of some polynomial with integer coefficients.
12. If d and p are metric functions on M, then so is $\sigma = \sqrt{d+p}$.
13. Let A be a subset of the metric space (M, d), then diam(A) = inf {d(x, y); x, y ∈ A}.
14. If **R** is equipped with the discrete metric, then diam (0, 4) = 4.
15. {1/n} is a Cauchy sequence.
16. Every convergent sequence is a Cauchy sequence.
17. Every Cauchy sequence is convergent.
18. A Cauchy sequence may not be bounded.
19. Every subsequence of a Cauchy sequence is Cauchy.
20. If a Cauchy sequence has a convergent subsequence, then the Cauchy sequence converges.
21. Equivalent metrics preserve Cauchy sequences. That is, if d and p are equivalent on M and $\{x_n\}$ is a sequence in M, then $\{x_n\}$ is Cauchy under the metric d if and only if $\{x_n\}$ is Cauchy under the metric p.
22. An arbitrary union of open sets is open.
23. An infinite intersection of open sets is never open.
24. An arbitrary intersection of closed sets is always closed.
25. A finite union of closed sets is always closed.
26. All sets are either open or closed.
27. If **R** is equipped with the discrete metric, the set (0, 1) is closed.

28. The set [0, 1) is neither open nor closed. [Hint: Be Careful!]
29. Let A be a subset of the metric space (M, d). If A has a limit point x in M, then there must be a sequence $\{a_n\}$ of elements of A that converges to x.
30. Let F be a subset of **R**, then the set of all limit points of F, $F'_{regular}$, under the regular metric is the same as F'_d — the set of all limit points of F under the metric $d(x, \ y) = \frac{|x-y|}{1+|x-y|}$.
31. Let A be a subset of M. If d and p are equivalent metrics, the set of all limit points of A in (M, d), A'_d, is identical to the set of all limit points of A in (M, p), A'_p. That is $A'_d = A'_p$.
32. Let A be a subset of (M, d). Then the set of all the limit points of A, A'_d, is a closed subset of M.
33. Every convergent sequence has a limit point.
34. No convergent sequence has more than one limit point.
35. The set of all limit points of a sequence may be uncountable.
36. Let M = (0, ∞) with the usual metric. Then {1/n} has a limit point.
37. A Cauchy sequence cannot have more than one limit point.
38. If a Cauchy sequence has a limit point, then it converges.
39. Let F be a closed subset of some metric space (M, d). If x is not a limit point of F, then x ∉ F.
40. An open set cannot have any limit points.
41. A finite set cannot have any limit points.
42. There exist metric spaces in which finite sets are not closed.
43. A closed set must contain the limits of all of its sequences.
44. If cl(A) = A, then A must be closed.
45. In an arbitrary metric space, any nonempty open set can be written as an (at most) countable union of disjoint open balls.
46. $\bigcap_{n=1}^{\infty}[\tan^{-1} n, \ \infty)$ is an open subset of **R**.
47. In any metric space, every nonempty perfect set is infinite.
48. In any metric space, every nonempty perfect set is uncountable.
49. Every dense set is perfect.
50. If P is a perfect subset of (M, d), then cl(P) = M.
51. **R** has a nonempty perfect subset that contains no rational numbers.
52. A nowhere dense set cannot have any limit points.
53. The compliment of a nowhere dense set is dense.
54. Suppose that *A* is a subset of some metric space (M, d) and that int(*A*) = ∅. Then A^c is a dense subset of M.
55. A discrete metric space has no proper dense subsets.
56. A discrete metric space does not have any nowhere dense subsets.

57. The compliment of a dense set is nowhere dense.