Solutions to H.W. #9

1. $\frac{\partial g}{\partial x}(x,y) = 2xy^3 - 3x^2y^2 \qquad \frac{\partial g}{\partial y}(x,y) = 3x^2y^2 - 2x^3y^2$

$\frac{\partial g}{\partial x}(-2,3) = -216 \qquad \frac{\partial g}{\partial y}(-2,3) = 156$

2. $\frac{\partial f}{\partial x}(x,y) = \cos(x+y) - (x-y)\sin(x+y)$

$\frac{\partial f}{\partial y}(x,y) = -\cos(x+y) - (x-y)\sin(x+y)$

$\frac{\partial f}{\partial x}\left(\frac{\pi}{2}, \frac{3\pi}{2}\right) = 1 \qquad \frac{\partial f}{\partial y}\left(\frac{\pi}{2}, \frac{3\pi}{2}\right) = -1$

3. $\frac{\partial f}{\partial x}(x,y) = ye^{xy} - ye^{x} \qquad \frac{\partial f}{\partial y}(x,y) = xe^{xy} - e^{x}$

4. $\frac{\partial h}{\partial x}(x,y) = -\frac{x}{(\sqrt{x^2+y^2})^3} \qquad \frac{\partial h}{\partial y}(x,y) = -\frac{y}{(\sqrt{x^2+y^2})^3}$

5. $f(x,y,z,w) = \ln\left(\frac{x+y}{z-w}\right) = \ln(x+y) - \ln(z-w)$

$\frac{\partial f}{\partial x} = \frac{1}{x+y} \qquad \frac{\partial f}{\partial y} = \frac{1}{x+y} \qquad \frac{\partial f}{\partial z} = \frac{-1}{z-w} \qquad \frac{\partial f}{\partial w} = \frac{1}{z-w}$

6. $\frac{\partial f}{\partial x_1} = \cos^{-1}(x_2x_3) \qquad \frac{\partial f}{\partial x_2} = \frac{-x_1x_3}{\sqrt{1-(x_2x_3)^2}} \qquad \frac{\partial f}{\partial x_3} = \frac{-x_1x_2}{\sqrt{1-(x_2x_3)^2}}$

$\frac{\partial f}{\partial x_4} = \frac{(1/x_5)}{1-\left(\frac{x_4}{x_5}\right)^2} = \frac{x_5}{x_5^2 - x_4^2} \qquad \frac{\partial f}{\partial x_5} = \frac{-(1/x_5)^2}{1+\left(\frac{x_4}{x_5}\right)^2} =$

$= \frac{-1}{x_5^2 + x_4^2}$

7. $\frac{\partial M}{\partial x_i} = a_i$

8. $\frac{\partial G}{\partial x_i} = \frac{2x_i}{2\sqrt{\sum_{i=1}^{n} x_i^2}} = \frac{x_i}{\|x\|}$

9. $P(V,T) = \frac{RT}{V}$

A small change in volume, ∂V, will produce a change in pressure, $\partial P\big|_{\partial V} = \frac{\partial P}{\partial V}\partial V = -\frac{RT}{V^2}\partial V$ (Thus, if ∂V is negative, i.e. if volume is decreased, pressure increases). Similarly, a small change in temperature produces a change in pressure $\partial P\big|_{\partial T} = \frac{\partial P}{\partial T}\partial T = \frac{R}{V}\partial T$.

If $|\partial V| = |\partial T|$, $\left|\frac{\partial P|_{\partial T}}{\partial P|_{\partial V}}\right| = \left|\frac{\frac{R}{V}\partial T}{-\frac{RT}{V^2}\partial V}\right| = \left|\frac{V}{T}\right|$.

when $V=1000$ and $T=600$ $\left|\frac{V}{T}\right| = \frac{1000}{600} = \frac{10}{6} > 1$

Hence $\left|\frac{\partial P|_{\partial T}}{\partial P|_{\partial V}}\right| > 1 \Rightarrow |\partial P|_{\partial T}| > |\partial P|_{\partial V}|$.

In other words, a small change of volume will have a lesser effect on pressure than a small change in temperature.

10. $R(r_1, r_2, r_3) = \frac{1}{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}}$

$$\frac{\partial R}{\partial r_1} = \frac{-1}{\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)^2} \cdot \frac{-1}{r_1^2} = \frac{1}{r_1^2}R.$$

Similarly $\frac{\partial R}{\partial r_2} = \frac{1}{r_2^2}R$ and $\frac{\partial R}{\partial r_3} = \frac{1}{r_3^2}R$

Since $r_1 = 10$ ohms, $r_2 = 15$ ohms, $r_3 = 20$ ohms

$\frac{\partial R}{\partial r_1} > \frac{\partial R}{\partial r_2} > \frac{\partial R}{\partial r_3}$. Hence a small change in r_1 will have the biggest effect on the effective resistance R.

11. $\frac{\partial u}{\partial x} = 12x^2y^4 + 2xy^5 + y$ $\qquad \frac{\partial u}{\partial y} = 16x^3y^3 + 5x^2y^4 + x$

$\frac{\partial^2 u}{\partial x^2} = 24xy^4 + 2y^5$ $\qquad \frac{\partial^2 u}{\partial y \partial x} = 48x^2y^3 + 10xy^4 + 1$

$\frac{\partial^2 u}{\partial x \partial y} = 48x^2y^3 + 10xy^4 + 1$ $\qquad \frac{\partial^2 u}{\partial y^2} = 48x^3y^2 + 20x^2y^3$

12. $\frac{\partial v}{\partial x} = \cos(x)\cos(y)$ $\qquad \frac{\partial v}{\partial y} = -\sin(x)\sin(y)$

$\frac{\partial^2 v}{\partial x^2} = -\sin(x)\cos(y)$ $\qquad \frac{\partial^2 v}{\partial y \partial x} = -\cos(x)\sin(y)$

$\frac{\partial^2 v}{\partial x \partial y} = -\cos(x)\sin(y)$ $\qquad \frac{\partial^2 v}{\partial y^2} = -\sin(x)\cos(y)$

13. $\frac{\partial w}{\partial x} = ye^{xy}$ $\qquad \frac{\partial w}{\partial y} = xe^{xy}$

$\frac{\partial^2 w}{\partial x^2} = y^2e^{xy}$ $\qquad \frac{\partial^2 w}{\partial y \partial x} = e^{xy} + xye^{xy}$

$\frac{\partial^2 w}{\partial x \partial y} = e^{xy} + xye^{xy}$ $\qquad \frac{\partial^2 w}{\partial y^2} = x^2e^{xy}$

14. $\frac{\partial f}{\partial x} = \tan^{-1}\left(\frac{x}{y}\right) + \frac{1/y}{1+\left(\frac{x}{y}\right)^2} = \tan^{-1}\left(\frac{x}{y}\right) + \frac{y}{x^2+y^2}$

$\frac{\partial f}{\partial y} = \frac{x\left(\frac{-x}{y^2}\right)}{1+\left(\frac{x}{y}\right)^2} = \frac{-x^2}{x^2+y^2}$

Thus, $\frac{\partial^2 f}{\partial x^2} = \frac{y}{x^2+y^2} - \frac{2xy}{(x^2+y^2)^2}$ $\qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{-x}{x^2+y^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{-x}{x^2+y^2} + \frac{x^2-y^2}{(x^2+y^2)^2} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{2xy}{(x^2+y^2)}$$

15. $Jf(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 8 & -7 \end{pmatrix}$. Thus

$Jf(1,-3) = \begin{pmatrix} 8 & -7 \end{pmatrix}$ and $Df(1,-3)(x,y) \equiv \begin{pmatrix} 8 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} =$

$= (8x-7y)$. In particular, $Df(1,-3)(x,y) = 8x-7y$.

16. $Jg(x,y) = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{x+y-(x-y)}{(x+y)^2} & -\frac{y+x-(y-x)}{(y+x)^2} \end{pmatrix}$

$= \begin{pmatrix} \frac{2y}{(x+y)^2} & -\frac{2x}{(x+y)^2} \end{pmatrix}$

$Jg(\frac{1}{2}, \frac{3}{2}) = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \end{pmatrix}$ and therefore $Dg(\frac{1}{2}, \frac{3}{2})(x,y) =$

$= \frac{3}{4}x - \frac{1}{4}y$.

17. $JP(x_1, \dots, x_6) = \begin{pmatrix} \frac{\partial P}{\partial x_1} & \dots & \frac{\partial P}{\partial x_6} \end{pmatrix}$ where

$\frac{\partial P}{\partial x_i} = \frac{2x_i}{x_4^2+x_5^2+x_6^2}$ when $i \in \{1,2,3\}$ and

$$\frac{\partial P}{\partial x_j} = \frac{-2x_j(x_1^2+x_2^2+x_3^2)}{(x_4^2+x_5^2+x_6^2)^2} = \frac{-2x_j}{x_4^2+x_5^2+x_6^2} P(x_1, \dots, x_6) \text{ when } j \in \{4,5,6\}$$

Since $P(3,4,5,1,1,-1) = \frac{50}{3}$ and since $x_4^2+x_5^2+x_6^2 = 3$

we see that $JP(3,4,5,1,1,-1) = \begin{pmatrix} \frac{6}{3} & \frac{8}{3} & \frac{10}{3} & \frac{-100}{9} & \frac{-100}{9} & \frac{100}{9} \end{pmatrix}$

Hence $Dp(3,4,5,1,1,-1)(x_1,x_2,x_3,x_4,x_5,x_6) = \frac{6}{3}x_1 + \frac{8}{3}x_2 + \frac{10}{3}x_3 - \frac{100}{9}x_4 - \frac{100}{9}x_5 + \frac{100}{9}x_6$

18.

$$JK(x,y,z) = \begin{pmatrix} \frac{\partial k_1}{\partial x} & \frac{\partial k_1}{\partial y} & \frac{\partial k_1}{\partial z} \\ \frac{\partial k_2}{\partial x} & \frac{\partial k_2}{\partial y} & \frac{\partial k_2}{\partial z} \\ \frac{\partial k_3}{\partial x} & \frac{\partial k_3}{\partial y} & \frac{\partial k_3}{\partial z} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -2y & 2z \\ 2x & 0 & -2z \\ -2x & 2y & 0 \end{pmatrix}\Bigg|_{(x,y,z)=(1,2,3)} = \begin{pmatrix} 0 & -4 & 6 \\ 2 & 0 & -6 \\ -2 & 4 & 0 \end{pmatrix}$$

Thus $Df(1,2,3)(x,y,z) \equiv \begin{pmatrix} 0 & -4 & 6 \\ 2 & 0 & -6 \\ -2 & 4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$

$$= \begin{pmatrix} -4y+6z \\ 2x-6z \\ -2x+4y \end{pmatrix}$$ or $Df(1,2,3)(x,y,z) = (-4y+6z, 2x-6z, -2x+4y)$

19. $$Jg(x,y) = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \\ \frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \\ \frac{y^2-x^2}{(x^2+y^2)^2} & -\frac{2xy}{(x^2+y^2)^2} \\ \frac{-2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{pmatrix}$$

$$Jg(1,1) = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ 0 & \frac{-1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$$ Hence $Dg(1,1)(x,y) \equiv \begin{pmatrix} -\frac{1}{2}x + \frac{1}{2}y \\ -\frac{1}{2}y \\ -\frac{1}{2}x \end{pmatrix}$

20. $Jv(t) = \begin{pmatrix} \frac{\partial v_1}{\partial t} \\ \frac{\partial v_2}{\partial t} \\ \frac{\partial v_3}{\partial t} \end{pmatrix}$. so $Jv(\frac{\pi}{6}) = \begin{pmatrix} \cos t \\ -2\sin 2t \\ 3\cos 3t \end{pmatrix}\Bigg|_{t=\frac{\pi}{6}} =$

$= \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{2\sqrt{3}}{2} \\ 0 \end{pmatrix}$ Hence $Dv(\frac{\pi}{6})(t) = \left(\frac{\sqrt{3}}{2}t, -\sqrt{3}t, 0\right)$
$= t\left(\frac{\sqrt{3}}{2}, -\sqrt{3}, 0\right)$

21. $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (4x+y)dx + (x+2y)dy$

22. $df = \frac{e^x}{2(e^x-y)}dx - \frac{1}{2(e^x-y)}dy$

23. Let $f(x,y) = (xe^y)^8 = x^8e^{8y}$. Then $f(0.99, 0.02) \approx$
$f(1,0) + Df(1,0)(0.99-1, 0.02-0) = 1 + 8(-0.01) + 8(0.02) =$
$= 1 + \frac{8}{100} = 1.08$

24. Let $f(x,y,z) = \sqrt{x^2+y^2+z^2} = \|(x,y,z)\|$. Then
$Jf(x,y,z) = \frac{2}{\|(x,y,z)\|}(x \ y \ z)$ $Jf(4,4,2) = \frac{2}{\sqrt{16+16+4}}(4 \ 4 \ 2)$
$= \frac{1}{\sqrt{8+1}}(4 \ 4 \ 2) = \frac{1}{3}(4 \ 4 \ 2)$.

$Df(4,4,2) = \frac{4}{3}x + \frac{4}{3}y + \frac{2}{3}z$

$f(4.01, 3.98, 2.02) \approx f(4,4,2) + Df(4,4,2)(4.01-4, 3.98-4, 2.02-2)$
$= 6 + \frac{4}{3}(0.01) - \frac{4}{3}(0.02) + \frac{2}{3}(0.02) = 6$

25. Let $V(x,y,z)$ be the volume of a box with dimensions $x, y,$ and z. Then $V(x,y,z) = xyz$.

Observe that the volume of gold needed to cover a box with dimensions $x, y,$ and z is the difference of the volumes of the box after it had been plated and the box before this job was complete.

$V(x+10^{-3}, y+10^{-3}, z+10^{-3}) - V(x,y,z) \approx dV$ where $dx = dy = dz = 10^{-3}$.

Observe that $dV = yz\,dx + xz\,dy + xy\,dz =$

$= (yz + xz + xy)\cdot 10^{-3}$. When $(x,y,z) = (1, 1.5, 2)$ this becomes

$(1.5 + 2 + 3)\cdot 10^{-3} = \underset{m}{5.5} \times \underset{mm}{10^{-3}}$ since 5.5 is in meters,

Vol of gold (in mm) is $5.5 \times 10^{3} \times 10^{-3} = 5.5$ mm.

26. The Volume of a cylinder with height h and radius r is $\pi r^2 h = V(h,r)$. Since the error in measurement of height is $\pm 0.002_{cm}$ and the error in radius is at most $\pm 0.001_{cm}$, the error in volume $\equiv |\Delta V| = |V_{actual} - V_{estimated}| \approx |dV| \leq \left|\frac{\partial V}{\partial h}\right||dh| + \left|\frac{\partial V}{\partial r}\right||dr| =$

$= \pi r^2 (0.002) + 2\pi r h (0.001) = 2\pi r (0.001)(r + h) =$

$= 2\pi (3.1)(0.001)(8.9)$

27. If f is linear then $Df(a)(x) = f(x)$. That is, f is its own derivative. To see this, observe that $\lim\limits_{h\to 0} \frac{\|f(a+h) - f(a) - f(h)\|}{\|h\|} = \lim\limits_{h\to 0} \frac{\|f(a) + f(h) - f(a) - f(h)\|}{\|h\|}$

$= \lim\limits_{h\to 0} \frac{\|0\|}{\|h\|} = 0.$

28. We wish to show that $\lim_{x \to a} f(x) = f(a)$. In other words, we would like to prove that for $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $\|f(x) - f(a)\| < \epsilon$ whenever $\|x-a\| < \delta$

Observe that $0 \le \|f(x) - f(a)\| = \|f(x) - f(a) - T(x-a) + T(x-a)\| \le$
$\le \|f(x) - f(a) - T(x-a)\| + \|T(x-a)\| = \dfrac{\|f(x)-f(a)-T(x-a)\|}{\|x-a\|}\|x-a\|$
$+ \|T(x-a)\|$ where T is the total derivative of f. (i.e. $T = Df(a)$).

Because T is linear, by excercise 6 of H.W#6, $\|T(x-a)\| \le$
$\le \|T\|\,\|x-a\|$.

Thus, we have that $\|f(x) - f(a)\| \le \dfrac{\|f(x)-f(a)-T(x-a)\|}{\|x-a\|}\|x-a\| +$
$+ \|T\|\,\|x-a\|$

Since $\lim_{x \to a} \dfrac{\|f(x)-f(a)-T(x-a)\|}{\|x-a\|} = 0$ choose δ_1 s.t. $\dfrac{\|f(x)-f(a)-T(x-a)\|}{\|x-a\|} < \dfrac{\epsilon}{2}$. Choose δ_2 s.t. $\|x-a\| < \dfrac{\epsilon}{2\|T\|}$ (ie $\delta_2 = \frac{\epsilon}{2\|T\|}$)

let $\delta(\epsilon) = \min\{\delta_1(\epsilon), \delta_2(\epsilon)\}$. Then $\|x-a\| < \delta(\epsilon) \Rightarrow$
$\dfrac{\|f(x)-f(a)-T(x-a)\|}{\|x-a\|}\|x-a\| < \dfrac{\|f(x)-f(a)-T(x-a)\|}{\|x-a\|} < \dfrac{\epsilon}{2}$

and $\|T\|\,\|x-a\| < \|T\| \dfrac{\epsilon}{2\|T\|} = \dfrac{\epsilon}{2}$

It follows that when $\|x-a\| < \delta(\epsilon)$, $\|f(x) - f(a)\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ as desired. In plane English, we have established that differentiability implies continuity.

29. If $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, then the component functions $f_1, f_2, \ldots, f_m$ must all be differentiable (why?). Thus it suffices to show that for $g: U \subset \mathbb{R}^n \to \mathbb{R}$ differentiable at $x=a$, the partials $\frac{\partial g}{\partial x_1}(a), \ldots, \frac{\partial g}{\partial x_n}(a)$ are defined. Let T be the total derivative of g that exists by hypothesis. Then $T: \mathbb{R}^n \to \mathbb{R}$ is a linear map that satisfies

$$\lim_{h \to 0} \frac{|g(a+h) - g(a) - T(h)|}{\|h\|} = 0.$$

Observe that $\displaystyle\lim_{\substack{t \to 0 \\ t \in \mathbb{R}}} \frac{|g(a+te_i) - g(a) - T(te_i)|}{\|te_i\|} =$

$$= \lim_{t \to 0} \frac{|g(a+te_i) - g(a) - tT(e_i)|}{|t|} = \lim_{t \to 0} \left| \frac{g(a+te_i) - g(a)}{t} - T(e_i) \right| =$$

(Since the limit along any direction must exist)

$\displaystyle\lim_{t \to 0} \left| \frac{g(a+te_i) - g(a)}{t} - T(e_i) \right| = 0$ implies that for any $\epsilon > 0$

$\exists\, \delta > 0$ s.t. $\displaystyle\left| \frac{g(a+te_i) - g(a)}{t} - T(e_i) \right| < \epsilon$ whenever $t \in B_\delta(0)$. This means that $T(e_i) - \epsilon < \dfrac{g(a+te_i) - g(a)}{t} <$

$< T(e_i) + \epsilon$. Since ϵ is arbitrary we see that $\displaystyle\lim_{t \to 0} \frac{g(a+te_i) - g(a)}{t} =$

$= T(e_i)$ but $\displaystyle\lim_{t \to 0} \frac{g(a+te_i) - g(a)}{t}$ is $\frac{\partial g}{\partial x_i}(a)$ by definition.

Thus $\frac{\partial g}{\partial x_i}(a)$ exists and is equal to the value of the total derivative

$Dg(a)(e_i)$ evaluated at e_i.

30. Observe that $\lim\limits_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} =$

$= \frac{1}{4} \lim\limits_{(x,y)\to(0,0)} \frac{4xy(x^2-y^2)}{x^2+y^2} = \frac{1}{4} \lim\limits_{r\to 0} \frac{2r^4 \sin 2\theta \cos 2\theta}{r^2} =$

$= \frac{1}{4} \lim\limits_{r\to 0} r^2 \sin 4\theta = 0$

Notice that $\frac{1}{4} \lim\limits_{r\to 0} \frac{r^2 \sin 4\theta}{r}$ also exists and equals 0.

but $\frac{1}{4} \lim\limits_{r\to 0} \frac{r^2 \sin 4\theta}{r} = \lim\limits_{(h,k)\to(0,0)} \frac{hk(h^2-k^2)}{(\sqrt{h^2+k^2})^3} =$

$= \lim\limits_{(h,k)\to(0,0)} \frac{\frac{hk(h^2-k^2)}{h^2+k^2} - 0 - T_0(h,k)}{\sqrt{h^2+k^2}} =$

$= \lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k) - f(0,0) - T_0(h,k)|}{\|(h,k)\|} = 0$

where $T_0(h,k) = 0$ (T_0 is the linear map from $\mathbb{R}^2$ to $\mathbb{R}$ that maps every point (h,k) to 0).

In particular, f is differentiable at $(0,0)$ with derivative $Df(0,0)(x,y) = 0$.

This means that $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ both exist with $\frac{\partial f}{\partial x}(0,0) = Df(0,0)(1,0) = 0$ and $Df(0,0)(0,1) = \frac{\partial f}{\partial y}(0,0) = 0$. (Problem 29)

To compute the mixed partials at $(0,0)$, observe that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(0,0)\right) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} =$$

$$= \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(h,0)}{h} = \lim_{h \to 0} \lim_{k \to 0} \frac{\frac{f(h,k) - f(h,0)}{k}}{h} =$$

$$= \lim_{h \to 0} \lim_{k \to 0} \frac{f(h,k)}{hk} = \lim_{h \to 0} \lim_{k \to 0} \frac{hk(h^2-k^2)}{hk(h^2+k^2)} =$$

$$= \lim_{h \to 0} \lim_{k \to 0} \frac{h^2-k^2}{h^2+k^2} = \lim_{h \to 0} \frac{h^2-0^2}{h^2+0^2} = 1$$

On the other hand, $\lim_{k \to 0} \lim_{h \to 0} \frac{h^2-k^2}{h^2+k^2} = \lim_{k \to 0} \frac{-k^2}{k^2} = -1$.

you should verify that $\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{k \to 0} \lim_{h \to 0} \frac{h^2-k^2}{h^2+k^2} = -1$.

The mixed partials are not equal because near $(0,0)$

$\frac{\partial^2 f}{\partial x \partial y}(h,k) = \frac{\partial^2 f}{\partial y \partial x}(h,k) \approx \frac{h^2-k^2}{h^2+k^2}$ with $\lim_{(h,k) \to (0,0)} \frac{\partial^2 f}{\partial x \partial y}(h,k) =$

$= \emptyset$ because $\lim_{(h,k) \to (0,0)} \frac{h^2-k^2}{h^2+k^2}$ is undefined.

Similarly, $\lim_{(h,k) \to (0,0)} \frac{\partial^2 f}{\partial y \partial x}(h,k) = \emptyset$. Thus, in particular,

the mixed partials $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are not continuous at $(0,0)$.