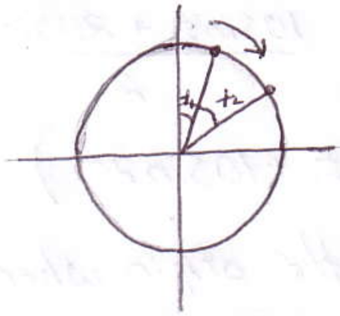
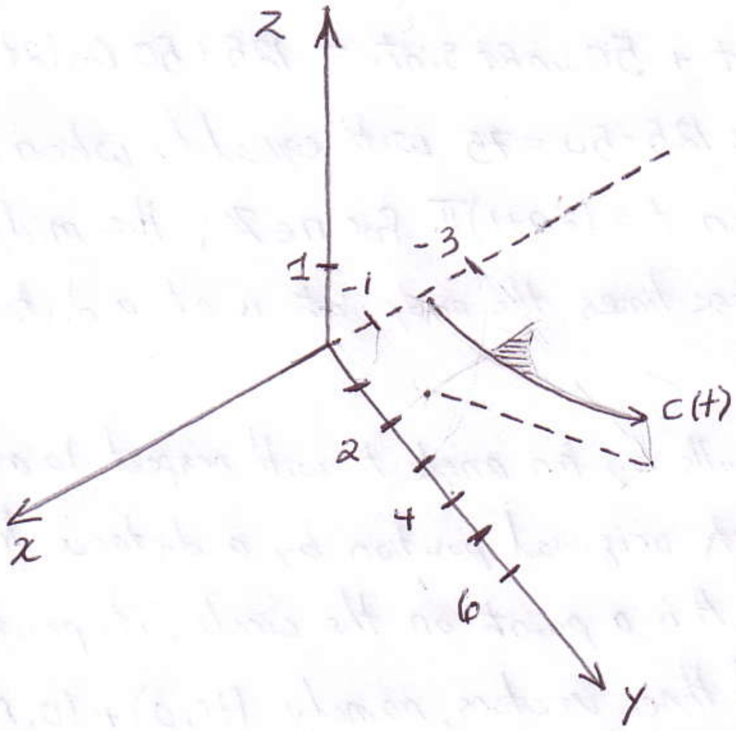
Solutions to H.W#8

1. This is the unit circle spanned by a particle that moves from the positive y-axis in the direction of the positive x-axis (i.e. clockwise)



2. This is the line traced from $(-1,2,0)$ in the direction $(2,1,1)$

3.



4. $c'(t) = 6i + 6tj + 3t^2 k = (6, 6t, 3t^2)$

5. $r'(t) = (4e^t, 24et^3, -\sin t)$

6. $l(t) = (0,3,-2) + t(1,2,10)$

7. $p(t) = (10\cos t, 10\sin t)$ $t \in [0,\pi]$

8. $c(t) = (\cos(-t), \frac{t}{2\pi}, \sin(-t)) = (\cos t, \frac{t}{2\pi}, -\sin t)$; $t \in [0, 4\pi]$

9. $\mathcal{E}(t) = (1, 4\cos t, 5\sin t)$ $t \in [0, 2\pi]$

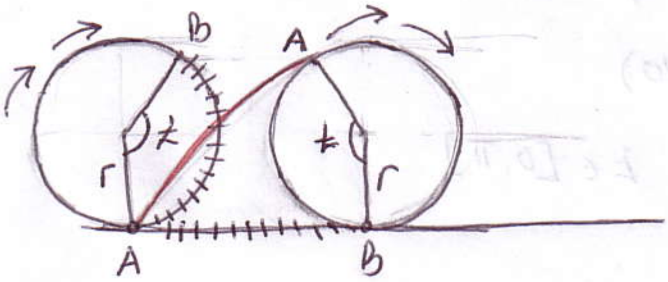
10. The midpoint is parameterized by

$$m(t) = \left(\frac{10\cos t + 20\cos 2t}{2}, \frac{10\sin t + 20\sin 2t}{2} \right) =$$

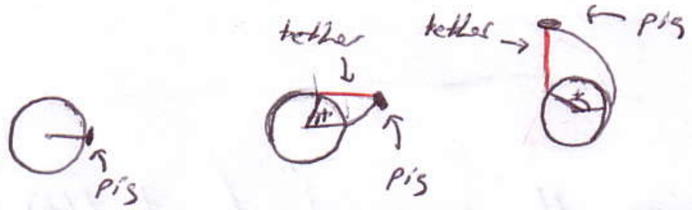
$$= (5\cos t + 10\cos 2t, 5\sin t + 10\sin 2t)$$

The midpoint is closest to the origin when $\|m(t)\|$ is as small as possible. This happens when $\|m(t)\|^2$ is as small as possible.
Let $H(t) = \|m(t)\|^2$. Then $H(t) = (5\cos t + 10\cos 2t)^2 + (5\sin t + 10\sin 2t)^2 =$
$= 125 + 50\cos 2t\cos t + 50\sin 2t\sin t = 125 + 50\cos(2t-t) = 125 + 50\cos t$
Observe that $H(t) \geq 125 - 50 = 75$ with equality when $\cos t = -1$.
It follows that when $t = (2n+1)\pi$ for $n \in \mathbb{Z}$, the midpoint is closest to the origin. At these times the midpoint is at a distance of $\sqrt{75} = 5\sqrt{5}$ from the origin.

11. If the circle rolls by an angle t with respect to a vertical line, it will displace from its original position by a distance of rt in the positive x-axis direction. If A is a point on the circle, its position may be described as a sum of three vectors, namely $(tr, 0) + (0, r) + (r\cos(-\frac{\pi}{2} - t), r\sin(\frac{-\pi}{2}-t$
$= (tr - r\sin t, r - r\cos(t))$ as you should surmize from the drawing below:
$\overset{\frown}{AB} = \overline{AB} = rt$



12. The pig, initially at the point $(r,0)$ will move counterclockwise to unwind the tether. Since the tether is always kept taut, we can obtain the position of the pig from the knowledge about the angle between the line segment connecting the center of the silo with the point at which the tether line touches the silo and the x-axis.



In particular, if the pig moved t radians about the silo, it must have uncoiled rt feet of tether. Its position can be obtained by moving away from $(r\cos t, r\sin t)$ in the direction opposite to the derivative at t a distance of rt feet. Hence position of pig at angle $t \equiv$
$p(t) = (r\cos t, r\sin t) + rt(\sin t, -\cos t) = (r\cos t + rt\sin t, r\sin t - rt\cos t)$
$t \in [0, \frac{L}{r}]$.

13. The equations can be written as $y = x+z-1$ and $2y = -5x+3z$.

Hence, we can replace y in the second equation by $x+z-1$ obtaining $2(x+z-1) = -5x+3z$. This equation implies that $z = 7x-2$.
Since $y = x+z-1$ it follows that $y = x+7x-2-1 = 8x-3$
The intersection is therefore the set of all points of the form $(x, 8x-3, 7x-2)$ which can be parametrized by $L(t) = (0,-3,-2) + t(1,8,7)$

14. let $x=\cos t$, $y=\sin t$ then $z=1-x-y=1-\cos t-\sin t$

Hence the intersection is parameterized by $p(t)=(\cos t, \sin t, 1-\cos t-\sin t)$ $t\in[0,2\pi]$.

15. $c'(t)=(2t-2, 6t, 4t+1)$ and $c'(1)=(0,6,5)$

At time $t>1$ the particle is on the line $L(t)=c(1)+(t-1)c'(1)=$

$=(4,7,3)+(t-1)(0,6,5)$.

At $t=2$, $L(2)=4,13,8)$.

To pass through the plane $z=23$, the z-coordinate at $L(t)$ must equal to 23; $3+(t-1)5=23$ $(t-1)5=20$ or $t-1=4$.

In other words, 4 units after $t=1$, the particle will have passed through the plane $z=23$.

16. When the child releases the stone, it travels in the direction tangent to the path along which the stone traveled during the swing.

Since $c(t)=(r\cos t, r\sin t)$, $\|c(t)\|=r$ is constant. Therefore the velocity vector $c'(t)$ is perpendicular to $c(t)$.

Suppose that she releases the stone at time t and that the stone hits (a,b). Then $(a,b)=c(t)+kc'(t)=(r\cos t, r\sin t)+kr(-\sin t, \cos t)$

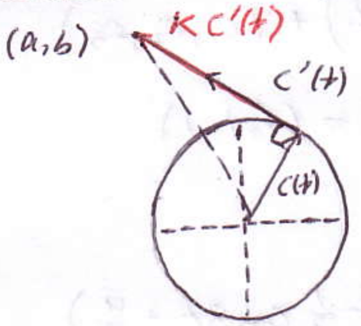
or $\left(\frac{a}{r},\frac{b}{r}\right)=(\cos t, \sin t)+k(-\sin t, \cos t)$. Since r is constant, we may write (α,β) instead of $\left(\frac{a}{r},\frac{b}{r}\right)$.

By taking the dot product of (α,β) with $(-\sin t, \cos t)$ we see that

$(\alpha,\beta)\cdot(-\sin t, \cos t)=(\cos t, \sin t)\cdot(-\sin t, \cos t)+k(-\sin t, \cos t)\cdot(-\sin t, \cos t)$

which reduces to $\beta \cos t - \alpha \sin t = k$. This can be written as $\sqrt{\alpha^2+\beta^2}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\cos t - \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\sin t\right) = k$. Without loss of generality, $\alpha > 0$ and, letting $\frac{\beta}{\sqrt{\alpha^2+\beta^2}} = \cos\left(\cos^{-1}\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right) = \cos\theta$ implies that $\frac{\alpha}{\sqrt{\alpha^2+\beta^2}} = \sin\theta$. This means that $k = \sqrt{\alpha^2+\beta^2}\left(\cos\theta\cos t - \sin\theta\sin t\right) =$
$= \sqrt{\alpha^2+\beta^2}\cos(t+\theta)$

Observe that, since $(a,b) = c(t) + k c'(t)$, $\|c(t)\| = \|c'(t)\| = r$, and $c(t)\cdot c'(t) = 0$, $\|(a,b)\|^2 = \|c(t)\|^2 + k^2\|c'(t)\|^2 = r^2 + r^2k^2$ as the picture below will help you visualize.



Hence $k = \frac{\sqrt{a^2+b^2-r^2}}{\sqrt{r^2}} = \sqrt{\alpha^2+\beta^2-1}$

Thus $\sqrt{\alpha^2+\beta^2-1} = \sqrt{\alpha^2+\beta^2}\cos(t+\theta)$ or $\sqrt{1-\frac{1}{\alpha^2+\beta^2}} = \cos(t+\theta)$

Thus $t = \cos^{-1}\left(\sqrt{1-\frac{1}{\alpha^2+\beta^2}}\right) - \theta = \cos^{-1}\sqrt{1-\frac{1}{\alpha^2+\beta^2}} - \cos^{-1}\frac{\beta}{\sqrt{\alpha^2+\beta^2}}$
$= \cos^{-1}\sqrt{\frac{a^2+b^2-r^2}{r^2}} - \cos^{-1}\frac{b}{\sqrt{a^2+b^2}}$ is the time at which the stone should be released.

17. $\int \left(te^t i + (e^{-5t}+1) j - \frac{e^{\sqrt{t}}}{\sqrt{t}} k\right) dt = \left(\int te^t dt, \int (e^{-5t}+1) dt, \int -\frac{e^{\sqrt{t}}}{\sqrt{t}} dt\right)$

$= \left(te^t - e^t + c_1, -\frac{1}{5}e^{-5t} + t + c_2, -2e^{\sqrt{t}} + c_3\right)$

18. $\int (\sin t \cos t\, i + \cos^3 t\, j - \sin^2 t\, k) dt = \left(\int \sin t \cos t\, dt, \int \cos^3 t\, dt, -\int \sin^2 t\, dt\right)$

$= \left(\frac{1}{2}\sin^2 t + c_1, -\sin t - \frac{1}{3}\sin^3 t + c_2, \frac{-1}{2}t + \frac{1}{4}\sin 2t + c_3\right)$

19, $r(t) = \int v(t)\,dt = \left(t^3 + t + c_1, t^2/2 + c_2, t^4/4 + c_3\right)$

Since $r(0) = (1,1,1)$ it follows that $(c_1, c_2, c_3) = (1,1,1)$.

Hence $r(t) = \left(t^3 + t + 1, t^2/2 + t, t^4/4 + 1\right)$

20, $r(t) = \left(-\cos t, \frac{1}{2}\sin 2t, -\frac{1}{3}\cos 3t\right) + (c_1, c_2, c_3)$

$r(0) = \left(-1, 0, -\frac{1}{3}\right) + (c_1, c_2, c_3) = (1,1,1)$

Hence $c_1 = 2$, $c_2 = 1$, $c_3 = \frac{4}{3}$

Thus $r(t) = \left(-\cos t + 2, \frac{1}{2}\sin 2t + 1, -\frac{1}{3}\cos 3t + \frac{4}{3}\right)$