Solutions to H.W #7

1. a) $\lim\limits_{(x,y)\to(1,-1)} \dfrac{x+4y}{1-x+y} = \dfrac{1-4}{1-1-1} = \dfrac{-3}{-1} = 3$

we can replace (x,y) by $(1,-1)$ because $\dfrac{x+4y}{1-x+y}$ is a rational function that is continuous at $(1,-1)$

b) $\lim\limits_{(x,y)\to(0,\frac{\pi}{2})} \dfrac{x+4y}{1-x+y} = \dfrac{2\pi}{1+\frac{\pi}{2}}$ by the same reasoning.

c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-e^{x^2+y^2}}{x^2+y^2} = \lim\limits_{(r,\theta)\to(0,0)} \dfrac{1-e^{(r\cos\theta)^2+(r\sin\theta)^2}}{(r\cos\theta)^2+(r\sin\theta)^2} =$

$= \lim\limits_{(r,\theta)\to(0,0)} \dfrac{1-e^{r^2}}{r^2} = \lim\limits_{u\to 0} \dfrac{1-e^u}{u} = -e^0 = -1$

d) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^3}{x^2+y^2+z^2} = 0$ because

$$0 \le \frac{|x|\,x^2}{x^2+y^2+z^2} \le \frac{|x|(x^2+y^2+z^2)}{x^2+y^2+z^2} = |x|$$

Since $\lim\limits_{(x,y,z)\to(0,0,0)} 0 = \lim\limits_{(x,y,z)\to(0,0,0)} |x| = 0$

the desired result follows.

e) $\lim_{(x,y)\to(0,0)} \frac{y^2}{\sqrt{x^2+y^2}} = 0$ because

$$0 \leq \frac{y^2}{\sqrt{x^2+y^2}} = \frac{|y||y|}{\sqrt{x^2+y^2}} \leq \frac{|y|\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = |y|$$

Since $\lim_{(x,y)\to(0,0)} 0 = \lim_{(x,y)\to(0,0)} |y| = 0$

Alternaternative solution:

$$\lim_{(x,y)\to(0,0)} \frac{y^2}{\sqrt{x^2+y^2}} = \lim_{(r,\theta)\to(0,\theta)} \frac{r^2\sin^2\theta}{r} = r\sin^2\theta = 0$$

2. a) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{1}{2}\frac{2xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{1}{2}\frac{r^2\sin 2\theta}{r} =$

$= \lim_{r\to 0} \frac{1}{2} r\sin 2\theta = 0$

b) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist:

$$\lim_{(x,y)\to(0,0)\,|\,y=0} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot 0}{x^2+0} = 0 \text{ while}$$

$$\lim_{(x,y)\to(0,0)\,|\,y=x} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}\lim_{x\to 0}\frac{x^2}{x^2} = \frac{1}{2}.$$

c) $\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{xy} = \lim_{u\to 0} \frac{1-\cos u}{u} = 0$

d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2 - x^2y}{(x^2+y^2)^{3/2}} = \lim\limits_{(x,y)\to(0,0)} \dfrac{xy(y-x)}{\left(\sqrt{(x^2+y^2)}\right)^3} =$

$= \lim\limits_{r\to 0} \dfrac{\frac{1}{2}r^2 \sin 2\theta \, r(\sin\theta - \cos\theta)}{r^3} = \lim\limits_{r\to 0} \frac{1}{2}\sin 2\theta(\sin\theta - \cos\theta)$

Since this limit depends on the value of θ, it does not exist.

Alternative solution:

$\lim\limits_{(x,y)\to(0,0)\big|_{x=0}} \dfrac{xy^2 - x^2y}{(x^2+y^2)^{3/2}} = \lim\limits_{y\to 0} \dfrac{0\cdot y^2 - 0^2 y}{(0^2+y^2)^{3/2}} = 0$ on the other hand

$\lim\limits_{(x,y)\to(0,0)\big|_{y=2x}} = \lim\limits_{x\to 0} \dfrac{4x^3 - 2x^3}{(x^2+4x^2)^{3/2}} = \lim\limits_{x\to 0} \dfrac{2x^3}{5x^3} = \dfrac{2}{5}.$

Hence the limit does not exist.

e) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y}{x^6+y^2}$ does not exist, because

$\lim\limits_{(x,y)\to(0,0)\big|_{x=0}} \dfrac{x^3y}{x^6+y^2} = 0$ while $\lim\limits_{(x,y)\to(0,0)\big|_{y=x^3}} \dfrac{x^3y}{x^6+y^2} =$

$= \lim\limits_{x\to 0} \dfrac{x^6}{x^6+x^6} = \dfrac{1}{2}$

3\. a) $$\lim_{(x,y,z)\to(0,0,0)} \frac{x^3+y^4+z^5}{x^2+y^2+z^2} = 0 \text{ because}$$

$$0 \le \frac{|x^3+y^4+z^5|}{x^2+y^2+z^2} \le \frac{|x|(x^2+y^2+z^2) + y^2(x^2+y^2+z^2) + |z|^3(x^2+y^2+z^2)}{x^2+y^2+z^2}$$

$$= |x| + y^2 + |z|^3 \to 0$$

b) $$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz^2}{x^4+y^4+z^4} \text{ does not exist because}$$

$$\lim_{(x,y,z)\to(0,0,0)\,\big|_{x=0}} \frac{xyz^2}{x^4+y^4+z^4} = 0 \text{ while}$$

$$\lim_{(x,y,z)\to(0,0,0)\,\big|_{x=y=z}} \frac{xyz^2}{x^4+y^4+z^4} = \lim_{x\to 0} \frac{x^4}{x^4+x^4+x^4} = \lim_{x\to 0} \frac{x^4}{3x^4} = \frac{1}{3}$$

c) $$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^3+z^4}{x^2+y^2+z^2} \text{ does not exist because}$$

$$\lim_{(x,y,z)\to(0,0,0)\,\big|_{y=z=0}} \frac{x^2+y^3+z^4}{x^2+y^2+z^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1 \text{ while}$$

$$\lim_{(x,y,z)\to(0,0,0)\,\big|_{x=z=0}} \frac{x^2+y^3+z^4}{x^2+y^2+z^2} = \lim_{y\to 0} \frac{y^3}{y^2} = 0$$

d) $\lim_{(x,y)\to(0,0)} \frac{x\cos y + y\sin y}{x^2+y^2}$ does not exist because

$$\lim_{(x,y)\to(0,0)}\Big|_{x=0} \frac{x\cos y + y\sin y}{x^2+y^2} = \lim_{y\to 0} \frac{y\sin y}{y^2} = 1$$

while

$$\lim_{(x,y)\to(0,0)}\Big|_{y=0} \frac{x\cos y + y\sin y}{x^2+y^2} = \lim_{x\to 0} \frac{x\cos(0) + 0\sin 0}{x^2+0^2} =$$

$$= \lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x} = \emptyset,$$

e) $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x} = \lim_{(x,y)\to(0,0)} y\frac{\sin xy}{xy} =$

$$= \lim_{(x,y)\to(0,0)} y \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} = 0\cdot \lim_{u\to 0} \frac{\sin u}{u}$$

$$= 0\cdot 1 = 0,$$

4. a) $\lim_{(x,y)\to(1,5)} 2x-5y = 2-25 = -23$

Suppose $|2x-5y-(-23)| = |2x-5y+23| = |2x-2-5(y-5)| =$

$= |2(x-1)-5(y-5)| = |(2,-5)\cdot(x-1,y-5)| \le \sqrt{2^2+(-5)^2}\,\|(x-1,y-5)\|$

$< \epsilon$ Thus $\|(x,y)-(1,5)\| < \frac{\epsilon}{\sqrt{29}} = \delta(\epsilon)$,

b) $\lim_{(x,y,z)\to(3,-1,0)} 4x+3y-z = 4\cdot 3+3(-1)-0 = L$

To prove that L is indeed the limit, we must show that $|4x+3y-z-L|<\epsilon$ whenever $\|(x,y,z)-(3,-1,0)\|<\delta$

Suppose $|4x+3y-z-4\cdot 3-3(-1)+0| = |4(x-3)+3(y+1)-z| =$

$= |(4,3,-1)\cdot(x-3,y+1,z)| \leq \|(4,3,-1)\|\,\|(x-3,y+1,z)\| < \epsilon$

$\|(x-3,y+1,z)\| < \frac{\epsilon}{\sqrt{16+9+1}} = \frac{\epsilon}{\sqrt{26}} = \delta(\epsilon).$

c) $\lim_{(x,y)\to(1,5)} (2x-5y,\ 4x+y) = (-23,9)$

To prove this observe that

$\|(2x-5y-23,\ 4x+y-9)\| = \|(2(x-1)-5(y-5),\ 4(x-1)+(y-5))\|$

$= \left\|\begin{pmatrix}2 & -5\\ 4 & 1\end{pmatrix}\begin{pmatrix}x-1\\ y-5\end{pmatrix}\right\| \leq \left\|\begin{pmatrix}2 & -5\\ 4 & 1\end{pmatrix}\right\| \|(x-1,y-5)\| < \epsilon$

$\|(x-1,y-5)\| < \frac{\epsilon}{\sqrt{4+25+16+1}} = \frac{\epsilon}{\sqrt{46}} = \delta(\epsilon).$