Solutions to H.W#6

1. a) $\begin{pmatrix} 3 & 5 \\ 0 & 1 \\ -1 & 4 \end{pmatrix}$ b) $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 4 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 3 & -\frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & 2 \end{pmatrix}$ d) $(1\ 1\ 1 \cdots 1)$
This is the 1×20 matrix with all of its entries equal to 1.

2. All linear maps $T: \mathbb{R}^n \to \mathbb{R}^m$ are of the form
$$T(x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^{n} a_{1i} x_i, \sum_{i=1}^{n} a_{2i} x_i, \ldots, \sum_{i=1}^{n} a_{mi} x_i \right)$$

a) linear map (its of the right form)

b) Not a linear map.

c) linear map.

3. a) $S: \mathbb{R}^2 \to \mathbb{R}^2$, $T: \mathbb{R}^2 \to \mathbb{R}^2$. Hence both ST & TS are defined.

$$ST(x,y) = S\left(\tfrac{1}{2}x + y, -\tfrac{1}{2}x + \tfrac{1}{5}y\right) = \Big(2\left(\tfrac{1}{2}x + y\right) + \left(-\tfrac{1}{2}x + \tfrac{1}{5}y\right),$$
$$-2\left(\tfrac{1}{2}x + y\right) + 5\left(-\tfrac{1}{2}x + \tfrac{1}{5}y\right)\Big) = \left(\tfrac{1}{2}x + \tfrac{11}{5}y, -\tfrac{7}{2}x - y\right)$$

$TS(x,y) = T(2x+y, -2x+5y) = \left(\frac{1}{2}(2x+y)+(-2x+5y),\right.$
$\left.-\frac{1}{2}(2x+y)+\frac{1}{5}(-2x+5y)\right) = \left(-x+\frac{11}{2}y, -\frac{7}{5}x+\frac{1}{2}y\right)$

b) $S: \mathbb{R}^3 \to \mathbb{R}^3$, $T: \mathbb{R}^3 \to \mathbb{R}^3$ Hence ST & TS are defined

$ST(x,y,z) = S(z,x,y) = (z+y, x, -y+x) = (y+z, x, x-y)$

$TS(x,y,z) = T(x+z, y, -z+y) = (-z+y, x+z, y)$

c) $S: \mathbb{R}^2 \to \mathbb{R}^4$, $T: \mathbb{R}^3 \to \mathbb{R}^2$ Hence ST is defined while TS is not defined.

$ST(x,y,\cancel{z}) = S(x,0) = (2x, x, 0, 4x)$

4. a) $\det\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} = 2-3 = -1 \neq 0$ Hence the matrix is invertible with inverse $\frac{1}{-1}\begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix}^T = -\begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} =$

$= \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$

b) $\det\begin{pmatrix} 4 & -1 \\ -8 & 2 \end{pmatrix} = 4\cdot 2-(-1)(-8) = 0$ Hence the matrix is not invertible.

(c) $\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0$ so the matrix is invertible with inverse $-1\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ notice that the matrix is its own inverse.

5. a) $\det\begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{pmatrix} = (-2)(-4)(-6) \neq 0$

Hence the matrix is invertible.
A little bit of thought will convince you that the inverse of a diagonal matrix $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ is the matrix $\begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix}$.

Hence the inverse matrix is $\begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{6} \end{pmatrix}$.

b) Observe that $\det\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = 1 \neq 0$ Hence the matrix is invertible. To find the inverse you may use the inverse theorem, which is rather tedious. Alternatively, observe that

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \mathcal{M}(T) \quad \text{where} \quad T(x,y,z) = (y,z,x).$$

This is a permutation $\begin{pmatrix} x & y & z \\ \downarrow & \downarrow & \downarrow \\ y & z & x \end{pmatrix}$. Its inverse will have to map $y \to x$, $z \to y$, $x \to z$. Hence $T^{-1}(x,y,z) =$
$= S(x,y,z) = (z,x,y)$.

To verify, observe that $TS(x,y,z) = T(z,x,y) = (x,y,z) =$
$= I(x,y,z)$. Similarly, $ST(x,y,z) = I(x,y,z)$

Now $\mathcal{M}(S) = \mathcal{M}(T^{-1}) = [\mathcal{M}(T)]^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ which gives the desired answer.

c) It would be extreemly tedious to find the inverse of $\begin{pmatrix} 1 & 3 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ using the inverse theorem directly.

Let's use a trick: By drawing red boxes arround the non-zero entries in the matrix above we see that this matrix, call it A, is of the form

$$A = \begin{pmatrix} B & O \\ O & C \end{pmatrix} \text{ where } B = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Finally, the matrix $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$:

$$A = \begin{pmatrix} \boxed{\begin{matrix} 1 & 3 \\ 1 & 2 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} \end{pmatrix}$$

Observe that for any matrix $D = \begin{pmatrix} E & O \\ O & F \end{pmatrix}$ where E and F are 2×2 matrices

$$AD = \begin{pmatrix} BE & O \\ O & CF \end{pmatrix} \text{ and } DA = \begin{pmatrix} EB & O \\ O & FC \end{pmatrix}$$

Thus $A^{-1} = \begin{pmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix}$. By the work done in question 4, we know that $B^{-1} = \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$ and $C^{-1} = C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Hence

$$A^{-1} = \begin{pmatrix} -2 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

6. a) $\left\| \begin{pmatrix} 3 & 1 \\ 7 & -2 \end{pmatrix} \right\| = \sqrt{3^2+1^2+7^2+(-2)^2}$

b) $\left\| \begin{pmatrix} 6 & 1 & -3 \\ 0 & 6 & 1 \\ 0 & 0 & 6 \end{pmatrix} \right\| = \sqrt{6^2+1^2+(-3)^2+6^2+1^2+6^2}$

c) $\left\| \begin{pmatrix} 1 & 1 & 0 & 4 \\ -4 & 0 & -1 & -1 \end{pmatrix} \right\| = \sqrt{1^2+1^2+4^2+(-4)^2+(-1)^2+(-1)^2}$

7. let A be $m \times n$ matrix with entries a_{ij}. Then for $x \in \mathbb{R}^n$

$$\|Ax^T\| = \left\|A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right\| = \left\|\left(\sum_{i=1}^n a_{1i}x_i, \ldots, \sum_{i=1}^n a_{mi}x_i\right)\right\| =$$

$$= \sqrt{\left|\sum_{i=1}^n a_{1i}x_i\right|^2 + \ldots + \left|\sum_{i=1}^n a_{mi}x_i\right|^2} \qquad (1)$$

let A_j denote the jth row of A. Observe that (1) can be written as

$$\sqrt{|A_1 \cdot x|^2 + \ldots + |A_m \cdot x|^2} \qquad (2)$$

by applying the Cauchy-Schwarz inequality to each $|A_j \cdot x|$ we get

$$\sqrt{|A_1 \cdot x|^2 + \ldots + |A_m \cdot x|^2} \leq \sqrt{(\|A_1\|\|x\|)^2 + \ldots + (\|A_m\|\|x\|)^2}$$

$$= \sqrt{(\|A_1\|^2 + \ldots + \|A_m\|^2)\|x\|^2} = \sqrt{\|A_1\|^2 + \ldots + \|A_m\|^2}\ \|x\|$$

$= \|A\|\|x\|$ and the result follows.

8. a) $T(2x+y, x+5y) \equiv \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Since $\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$ is invertible ($\det \neq 0$), the linear map T is invertible.

The inverse of $\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$ is $\frac{1}{9}\begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}^T = \frac{1}{9}\begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}$

so $T^{-1}(x,y) = \frac{1}{9}(5x - y, -x + 2y)$.

b) $T(x,y) = (2x - 3y, -4x + 6y) \equiv \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Since $\begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}$ is not invertible, neither is T.

c) $T(x_1, x_2, x_3, \ldots) = (10x_1, 9x_2, 8x_3, \ldots x_{10})$

$$\equiv \begin{pmatrix} 10 & 0 & 0 & \cdots & 0 \\ 0 & 9 & 0 & & \vdots \\ \vdots & \vdots & 8 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & & & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{10} \end{pmatrix}$$

Since the matrix is diagonal, its determinant is $10 \cdot 9 \cdot 8 \cdot \ldots \cdot 1 = 10! \neq 0$ Hence the matrix is invertible.

with inverse $\begin{pmatrix} \frac{1}{10} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{9} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{8} & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$ Hence

$$T^{-1}(x_1, x_2, x_3, \ldots, x_{10}) = \left(\tfrac{1}{10}x_1, \tfrac{1}{9}x_2, \tfrac{1}{8}x_3, \ldots, x_{10}\right)$$

d) $T(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4, 0)$ is not invertible because it is not 1-1: $T(1,1,1,1) = T(2,1,1,1)$.

Functions that are not 1-1 cannot have an inverse.

9. a) $T(x,y) = (-x, y)$ $\quad \mathcal{M}(T) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

b) $T_{\frac{\pi}{2}}(x,y)$ is the desired map.

Since $T_{\frac{\pi}{2}}(1,0) = (0,1)$ & $T_{\frac{\pi}{2}}(0,1) = (-1,0)$

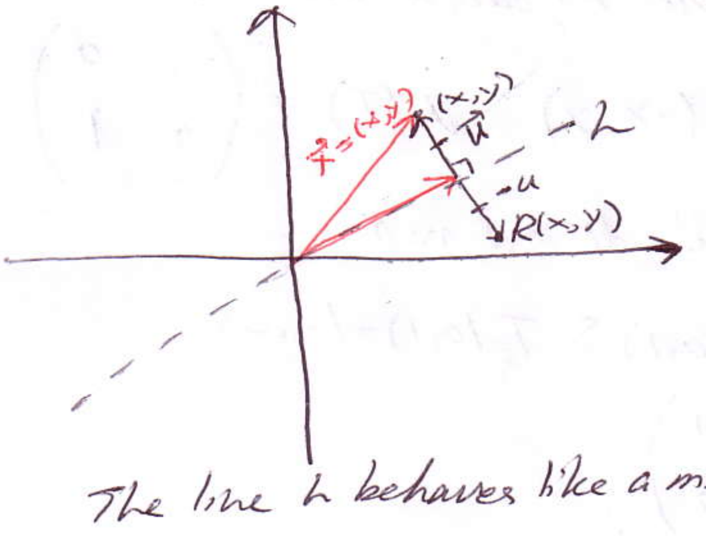
$$\mathcal{M}(T_{\frac{\pi}{2}}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

c) the line $ax+by=0$ is the set $\{(x,y): ax+by=0\} = L$
The projection onto this line is the same as projection onto the vector that generates this line.
let $w=(b,-a)$ then L is the range of $S(t)=t(b,-a)$
Hence, the projection onto the line $ax+by=0$ is the linear map $P_w(x,y) = \frac{w\cdot(x,y)}{\|w\|^2}w = \frac{bx-ay}{(\sqrt{a^2+b^2})^2}(b,-a)$

$$M(P_w(x,y)) = \begin{pmatrix} \frac{b^2}{(\sqrt{a^2+b^2})^2} & \frac{-ab}{(\sqrt{a^2+b^2})^2} \\ \frac{-ab}{(\sqrt{a^2+b^2})^2} & \frac{a^2}{(\sqrt{a^2+b^2})^2} \end{pmatrix} =$$

$$= \frac{1}{(\sqrt{a^2+b^2})^2}\begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} = \frac{1}{a^2+b^2}\begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix}$$

d) let $L=\{(x,y): ax+by=0\}$ be any line through the origin. let $R(x,y)$ be the reflection through this line. The picture bellow illustrates the geometric significance of R.



The line L behaves like a mirror.

This picture suggests that $R(x,y) = P_\omega(x,y) - \vec{u}$, where ω is any vector that generates the line $ax+by=0$ and $\vec{u}$ is the displacement from the arrow head of $P_\omega(x,y)$ to (x,y). In particular, $\vec{u} = (x,y) - P_\omega(x,y)$

Hence $R(x,y) = P_\omega(x,y) - \vec{u} = P_\omega(x,y) - \big((x,y) - P_\omega(x,y)\big)$

$$= 2P_\omega(x,y) - (x,y) =$$

$$= 2\,\frac{bx-ay}{a^2+b^2}(b,-a) - (x,y) =$$

$$= \left(\frac{2b^2x - 2aby}{a^2+b^2}, \frac{-2abx+2a^2y}{a^2+b^2}\right) - (x,y) =$$

$$= \left(\frac{2b^2x - (a^2+b^2)x - 2aby}{a^2+b^2}, \frac{-2abx + 2a^2y - (a^2+b^2)y}{a^2+b^2}\right) =$$

$$= \left(\frac{(b^2-a^2)x - 2aby}{a^2+b^2}, \frac{-2abx - (b^2-a^2)y}{a^2+b^2}\right) =$$

$$= \frac{1}{a^2+b^2}\left((b^2-a^2)x - 2aby, -2abx - (b^2-a^2)y\right)$$

Finally, we wish to rotate by the angle $\frac{\pi}{2}$. That is the desired map is $T_{\frac{\pi}{2}}R(x,y) = T_{\frac{\pi}{2}}(R(x,y)) = T_{\frac{\pi}{2}}\left(\frac{1}{a^2+b^2}([b^2-a^2]x - 2aby,\right.$
$\left.-2abx - [b^2-a^2]y\right)$ (1)

Since $T_{\frac{\pi}{2}}(x,y) = (-y,x)$,

$$(1) = \left(2abx + (b^2-a^2)y,\ -(b^2-a^2)x + 2aby\right)\frac{1}{a^2+b^2}$$

Since the particular line that we were given is

$4x+5y=0$

$$T_{\frac{\pi}{2}}R(x,y) = \frac{1}{4^2+5^2}\left(2\cdot4\cdot5x + (25-16)y,\ -(25-16)x + 2\cdot4\cdot5y\right)$$

$$= \frac{1}{31}\left(40x+9x,\ -9x+40y\right) \equiv \frac{1}{31}\begin{pmatrix}40 & 9\\ -9 & 40\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

10. a) $T(x,y,z) = (-x,y,z) \equiv \begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$

b) $T(x,y,z) = (0,y,z) \equiv \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$

c) $T(x,y,z) = (-x,-y,-z) \equiv \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$

d) $T(x,y,z) = (x,0,0)-(0,y,z) = (x,-y,-z) \equiv$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

11. If the particle's position is initially (x, y, z), then, in cylindrical coordinates, this position is given as $(r\cos\theta, r\sin\theta, z)$ where $r = \sqrt{x^2+y^2}$ and θ is the angle between (x,y) and the x-axis.

After T time units, the particle will have rotated $T\omega$ radians and its new position is given by $(r\cos(\theta+T\omega), r\sin(\theta+T\omega), z) =$

$$= \left(r\cos\theta\cos(T\omega) - r\sin\theta\sin(T\omega),\ r\sin\theta\cos(T\omega) + r\cos\theta\sin(T\omega),\ z\right)$$

$$= \left(\cos(T\omega)x - \sin(T\omega)y,\ \sin(T\omega)x + \cos(T\omega)y,\ z\right) = R_T(x,y,z)$$

In other words $R_T(x,y,z) \equiv \begin{pmatrix} \cos(T\omega) & -\sin(T\omega) & 0 \\ \sin(T\omega) & \cos(T\omega) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

which is a linear map.