# Solutions to HW#18

1. $f(x,y) = 8x + 3y$ ; $(x-1)^2 + (y+2)^2 = 9$.

$\nabla f = (8,3) \qquad \nabla g = 2(x-1,\ y+2)$

$$(8,3) = 2\lambda(x-1,\ y+2)$$

$$8 = 2\lambda(x-1) \qquad (1)$$

$$3 = 2\lambda(y+2) \qquad (2)$$

$$(x-1)^2 + (y+2)^2 = 9 \qquad (3)$$

Equations (1) and (2) imply that $\lambda \neq 0$

Notice that $\left(\frac{8}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = (x-1)^2 + (y+2)^2 = 9$.

Thus $4\lambda^2 = \frac{73}{9}$ so $\lambda = \pm\frac{\sqrt{73}}{6}$ and $2\lambda = \pm\frac{\sqrt{73}}{3}$

setting $2\lambda = \frac{\sqrt{73}}{3}$ in equations (1) and (2) we obtain

$x = \frac{24}{\sqrt{73}} + 1$ and $y = \frac{9}{\sqrt{73}} - 2$ or $\left(\frac{24}{\sqrt{73}}+1,\ \frac{9}{\sqrt{73}}-2\right)$.

setting $2\lambda = -\frac{\sqrt{73}}{3}$, we obtain $\left(\frac{-24}{\sqrt{73}}+1,\ \frac{-9}{\sqrt{73}}-2\right)$

clearly the maximum value of $f$ is $f\left(\frac{24}{\sqrt{73}}+1,\ \frac{9}{\sqrt{73}}-2\right)$

and the minimum is $f\left(-\frac{24}{\sqrt{73}}+1,\ \frac{-9}{\sqrt{73}}-2\right)$.

2. $f(x,y) = xy$ ; $\frac{x^2}{25} + \frac{y^2}{4} = 1$.

The constraint is an ellipse. It is therefore closed and bounded.

Let $c(t) = (5\cos t, 2\sin t)$ be a parametrization of the constraint. Then $f(c(t)) = 10\sin t\cos t = 5\sin 2t$.
The restricted function $f$ attains its maximum value when it is equal to 5. at $t = \frac{\pi}{4}$ and $t = \frac{\pi}{4} + \pi$.
It attains its minimum value when it's equal to $-5$. at $t = \frac{3\pi}{4}$ and $t = \frac{3\pi}{4} + \pi$.
The critical points are therefore

$$\left.\begin{array}{l} \left(\frac{5\sqrt{2}}{2}, \frac{2\sqrt{2}}{2}\right) \\ \left(-\frac{5\sqrt{2}}{2}, -\frac{2\sqrt{2}}{2}\right) \end{array}\right\} \text{max}$$

$$\left.\begin{array}{l} \left(\frac{-5\sqrt{2}}{2}, \frac{2\sqrt{2}}{2}\right) \\ \left(\frac{5\sqrt{2}}{2}, -\frac{2\sqrt{2}}{2}\right) \end{array}\right\} \text{min.}$$

3. $f(x,y) = 8x + 27y$; $x^4 + y^4 = 1$

$\nabla f = (8, 27)$ $\nabla g = (4x^3, 4y^3)$

$$8 = 4\lambda x^3 \qquad (1)$$

$$27 = 4\lambda y^3 \qquad (2)$$

$$x^4 + y^4 = 1 \qquad (3)$$

Equations (1) and (2) imply that

$$2 = (4\lambda)^{1/3} x \qquad (4)$$

$$3 = (4\lambda)^{1/3} y \qquad (5)$$

Solving for $x$ and $y$ and substituting into equation (3) we obtain

$$\left(\frac{2}{(4\lambda)^{1/3}}\right)^4 + \left(\frac{3}{(4\lambda)^{1/3}}\right)^4 = 1$$

Thus $4\lambda = \pm (97)^{3/4}$

Letting $4\lambda = (97)^{3/4}$, we obtain the critical point $\left(\frac{2}{(97)^{1/4}}, \frac{3}{(97)^{1/4}}\right)$. Setting $4\lambda = -(97)^{3/4}$ we obtain the critical point $\left(\frac{-2}{(97)^{1/4}}, \frac{-3}{(97)^{1/4}}\right)$.

Notice that $f\left(\frac{2}{(97)^{1/4}}, \frac{3}{(97)^{1/4}}\right) = \frac{97}{(97)^{1/4}} = (97)^{3/4}$.

Similarly, $f\left(\frac{-2}{(97)^{1/4}}, \frac{-3}{(97)^{1/4}}\right) = -(97)^{3/4}$

Thus the maximum value under the constraint is $(97)^{3/4}$ and the min value of $f$ under the constraint is $-(97)^{3/4}$.

4. $f(x,y) = x^2 + y^3$; $x^2 + y^2 = 49$

$\nabla f = (2x, 3y^2)$ $\qquad \nabla g = (2x, 2y)$

$$2x = 2\lambda x \qquad (1)$$

$$3y^2 = 2\lambda y \qquad (2)$$

$$x^2 + y^2 = 49 \qquad (3)$$

If $x=0$, equation (3) implies that $y=\pm 7$. If $y=0$, equation (2) implies that $x=\pm 7$.

Suppose $x\neq 0$ & $y\neq 0$

Then equations (1) and (2) can be simplified to

$$2=2\lambda \quad (4)$$

$$3y=2\lambda \quad (5)$$

Hence $\lambda=1$ and $y=\frac{2}{3}$. Substituting this value of $y$ into equation (3) yields $x=\pm\sqrt{49-\frac{4}{9}}$.

Thus, the critical points are

$$(0,\pm 7)$$

$$(\pm 7, 0)$$

$$\left(\pm\sqrt{49-\tfrac{4}{9}}, \tfrac{2}{3}\right).$$

Now $f(0,7)=343$, $f(0,-7)=-343$,

$f(\pm 7,0)=49$, and $f\left(\pm\sqrt{49-\frac{4}{9}},\frac{2}{3}\right)=49-\frac{4}{9}+\frac{8}{27}$

Thus the maximum value is 343 and the minimum is $-343$.

5. $f(x,y,z) = xyz$; $x^2+y^2+z^2 = a^2$

$\nabla f = (yz,\ xz,\ xy)$ $\quad \nabla g = (2x,\ 2y,\ 2z)$

Thus we must solve the system of equations

$$yz = 2\lambda x \quad (1)$$
$$xz = 2\lambda y \quad (2)$$
$$xz = 2\lambda z \quad (3)$$
$$x^2+y^2+z^2 = a^2 \quad (4)$$

Multiplying (1) by $x$, (2) by $y$, (3) by $z$ and adding yields

$$3(xyz) = 2\lambda(x^2+y^2+z^2) = 2\lambda a^2$$

If $\lambda = 0$ then $f(x,y,z) = xyz = \frac{2}{3}0 \cdot a^2 = 0$. Assume $\lambda \neq 0$. Then $x \neq 0, y \neq 0, z \neq 0$. Notice that (1) can then be written as $z = 2\lambda \frac{x}{y}$ and (2) can be written as $z = 2\lambda \frac{y}{x}$. Setting the two equations equal to each other, we obtain $x^2 = y^2$. Similarly, you shoud show that $x^2 = z^2$. Letting $x^2 = y^2 = z^2$ in the fourth equation we get $x^2+x^2+x^2 = a^2$ or $x^2 = \frac{a^2}{3}$

Thus, the critical points are $\left(\pm\frac{a}{\sqrt{3}}, \pm\frac{a}{\sqrt{3}}, \pm\frac{a}{\sqrt{3}}\right)$.

The maximum value is therefore $\frac{a^3}{\sqrt{27}}$ and the minimum value is $-\frac{a^3}{\sqrt{27}}$.

6. $f(x,y,z) = z - x^2 - y^2$; $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$.

$\nabla f = (-2x, -2y, 1)$ $\quad \nabla g = \left(\frac{2}{4}x, \frac{2}{9}y, \frac{2}{16}z\right)$

We must solve the system of equations

$$-2x = 2\lambda \frac{x}{4} \qquad (1)$$

$$-2y = 2\lambda \frac{y}{9} \qquad (2)$$

$$1 = 2\lambda \frac{z}{16} \qquad (3)$$

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1 \qquad (4)$$

Observe first that (3) implies that $\lambda \neq 0$ and $z = \frac{8}{\lambda}$. Multiplying (1) by $x$, (2) by $y$ and (3) by $z$ and then adding yields

$$-2x^2 - 2y^2 + 1 = 2\lambda\left(\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16}\right) = 2\lambda \quad (5)$$

If $x = y = 0$, $z = \pm 4$ giving us the critical points $(0, 0, \pm 4)$.

Assume $x \neq 0$. Dividing both sides of (1) by $2x$ yields

$-1 = \frac{\lambda}{4}$ or $\lambda = -4$. Since $z = \frac{8}{\lambda}$, it follows that $z = -2$. Substituting the value of $\lambda$ into (2) gives $-2y = -\frac{8}{9}y$ or $y = 0$. Substituting $y=0$ and $z = -2$ into equation (4) gives $\frac{x^2}{4} + \frac{1}{4} = 1 \quad x^2 + 1 = 4$ or $x = \pm\sqrt{3}$. We thus obtain the critical points

$(\pm\sqrt{3}, 0, -2)$.

Lastly, assume $y \neq 0$. Then dividing equation (2) by $2y$ yields $-1 = \frac{\lambda}{9}$ or $\lambda = -9$. Thus $z = \frac{8}{-9}$. Substituting $\lambda = -9$ into (1) we get $-2x = -18\frac{x}{4}$ or $x = 0$. Setting $x = 0$ and $z = -\frac{8}{9}$ in (4) yields $\frac{y^2}{9} + \frac{8^2}{9^2 \cdot 16} = 1$ or $y^2 = 9 - \frac{4}{9} = \frac{77}{9}$ Hence $(0, \pm\frac{\sqrt{77}}{3}, -\frac{8}{9})$ is another pair of critical points.

Notice that $f(0,0,-4) = -4$, $f(0,0,4) = 4$, $f(\pm\sqrt{3}, 0, -2) = -2 - 3 = -5$, and $f(0, \pm\frac{\sqrt{77}}{3}, -\frac{8}{9}) =$
$= \frac{8}{9} - \frac{77}{9} = -\frac{69}{9}$

Hence the minimum value is attained at the points $(0, \pm\frac{\sqrt{77}}{3}, -\frac{8}{9})$ and the maximum value is attained at $(0,0,4)$.

7. $f(x,y,z) = x^2+y^2+z^2$; $x^4+y^4+z^4 = 1$

$\nabla f = (2x, 2y, 2z)$ $\quad \nabla g = (4x^3, 4y^3, 4z^3)$

Our system of equations is therefore

$$2x = 4\lambda x^3 \qquad (1)$$

$$2y = 4\lambda y^3 \qquad (2)$$

$$2z = 4\lambda z^3 \qquad (3)$$

$$x^4+y^4+z^4 = 1 \qquad (4)$$

If $x=y=0$, by equation (4) $z=\pm1$, yielding the critical point $(0,0,\pm1)$. By symmetry $(\pm1,0,0)$, and $(0,\pm1,0)$ are also critical points.

If $x=0$, but $y\neq 0$ and $z\neq 0$ we get from eq. (2) & (3) that

$$1 = 2\lambda y^2 \qquad (5)$$

$$1 = 2\lambda z^2 \qquad (6)$$

These equations imply that $\lambda \neq 0$. Furthermore $y^2=z^2$, therefore $y^4=z^4$. Substituting $y^4=z^4$ into equation (4) yields $2y^4=1$ or $y=\pm 2^{-1/4}$. We thus have the critical points $(0,\pm2^{-1/4},\pm2^{-1/4})$. By symmetry, $(\pm2^{-1/4},0,\pm2^{-1/4})$ and $(\pm2^{-1/4},\pm2^{-1/4},0)$ are also critical points.

Finally, equations (1)-(3) imply that $x^2=y^2=z^2$ whenever $x\neq0$, $y\neq0$, and $z\neq0$. Thus, by eq. (4) $3x^4=1$. $(\pm3^{-1/4},\pm3^{-1/4},\pm3^{-1/4})$ are therefore the remaining critical points.

Now $f(0,0,\pm1)=f(0,\pm1,0)=f(\pm1,0,0)=1$

$$f(0,\pm2^{-1/4},\pm2^{-1/4}) = f(\pm2^{-1/4},0,\pm2^{-1/4}) = f(\pm2^{-1/4},\pm2^{-1/4},0) =$$

$$= 2\sqrt{2}$$

and $f(\pm3^{-1/4},\pm3^{-1/4},\pm3^{-1/4}) = 3\sqrt{3}$. The min is 1 and the max is $3\sqrt{3}$.

8. $f(x,y) = x^2 + y^2$; $5x - 9y = 1$

We can locate the critical points by either parametrizing the constraint and thereby reducing to single variable calculus or by using Lagrange multipliers. The Lagrange multiplier method yields the equations

$$2x = 5\lambda \qquad (1)$$

$$2y = -9\lambda \qquad (2)$$

$$5x - 9y = 1 \qquad (3)$$

Multiplying (1) by 5 & (2) by $-9$ and adding gives

$$2 = 2(5x-9y) = (25+81)\lambda. \quad \text{Thus } \lambda = \frac{1}{106}.$$

Substituting this value of $\lambda$ into (1) and (2) gives $x = \frac{5}{212}$ and $y = \frac{-9}{212}$. Thus the only critical point is $\left(\frac{5}{212}, \frac{-9}{212}\right)$ and it corresponds to $\lambda = \frac{1}{106}$.

We can use the second derivative test to classify this critical point. Since $Hg = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $Hf = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$,
$\omega\left(\left(\frac{5}{212}, \frac{-9}{212}\right), \frac{1}{106}\right) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ which is positive definite. Hence $f\left(\frac{5}{212}, \frac{-9}{212}\right)$ is a local minimum.

9. $f(x,y) = -x^2 - y^2; \quad x = y^2$

Let $c(t) = (t^2, t)$. We wish to find the critical points of $p(t) = f(c(t)) = -(t^2)^2 - t^2 = -t^4 - t^2$.

Observe that $p(t) \leq 0$ for all $t$, with equality if and only if $t=0$.

Thus the only critical point is $(0,0)$. At this point $f$ has a maximum.

10. $f(x,y) = (x^3 - x)y^2; \; 2x + 3y = 0$

We could try the direct method by letting $c(t) = (-3t, 2t)$ for instance. We then would obtain the function

$$p(t) = \left(-27t^3 + 3t\right) 4t^2$$

Since this is a polynomial of degree 5, it would not be so easy to find the critical points in terms of $t$.

Let us try the Lagrange multiplier method instead.

$\nabla f = \left((3x^2 - 1)y^2, \; 2(x^3 - x)y\right) \quad \nabla g = (2, 3)$.

We obtain the equations below

$$(3x^2 - 1)y^2 = 2\lambda \qquad (1)$$

$$2(x^3 - x)y = 3\lambda \qquad (2)$$

$$2x + 3y = 0 \qquad (3)$$

Observe that $x=0$ iff $y=0$ and that $(0,0)$ satisfies equations (1) (2), and (3) when $\lambda=0$. Thus $(0,0)$ is our first critical point.

Suppose that $x\neq 0$. Then $y\neq 0$. We can rewrite equation (1) as $\frac{1}{2}(3x^2-1)y^2=\lambda$, equation (2) as $\frac{2}{3}(x^3-x)y=\lambda$, and equation (3) as $y=-\frac{2}{3}x$.

Observe that

$$\frac{1}{2}(3x^2-1)y^2=\frac{2}{3}(x^3-x)y$$

Upon dividing by $y$, we obtain

$$\frac{1}{2}(3x^2-1)y=\frac{2}{3}(x^3-x)=-\frac{2}{3}x(1-x^2)$$

Setting $y=-\frac{2}{3}x$ yields

$$\frac{1}{2}(3x^2-1)\left(-\frac{2}{3}x\right)=\left(-\frac{2}{3}x\right)(1-x^2)$$

which reduces to

$$\frac{1}{2}(3x^2-1)=1-x^2$$

Since $x\neq 0$.

Thus $3x^2-1=2-2x^2$ or $x=\pm\sqrt{\frac{3}{5}}$

and $y=\mp\frac{2}{3}\sqrt{\frac{3}{5}}$, giving us the critical points $\sqrt{\frac{3}{5}}\left(1,-\frac{2}{3}\right)$ and $\sqrt{\frac{3}{5}}\left(-1,\frac{2}{3}\right)$

Since $Hg = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\omega(a,\lambda) = Hf = \begin{pmatrix} 6xy^2 & 6x^2y-2y \\ 6x^2y-2y & 2(x^3-x) \end{pmatrix}$

At $(0,0)$ $Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and the second derivative test gives no information

However, we can write $f|_g$ as $(x^3-x)\left(\frac{2}{3}x\right)^2$.

When $-1<x<0$, $x^3-x>0$, while if $0<x<1$ $x^3-x<0$.

It follows that $(0,0)$ is a saddle point.

You should check what happens with the remaining two points.

11. $f(x,y,z) = 2x^2+y^2+4z^2$; $x+y+z=1$

$$\nabla f = (4x, 2y, 8z) \qquad \nabla g = (1,1,1)$$

$$4x = \lambda \qquad (1)$$

$$2y = \lambda \qquad (2)$$

$$8z = \lambda \qquad (3)$$

$$x+y+z=1 \qquad (4)$$

Solving for $x$, $y$, and $z$ in terms of $\lambda$ and plugging the results into equation (4), we obtain

$$\frac{\lambda}{4} + \frac{\lambda}{2} + \frac{\lambda}{8} = 1$$

so $\lambda = \frac{8}{7}$

Hence, equations (1)–(3) imply that $x = \frac{2}{7}$, $y = \frac{4}{7}$, and $z = \frac{1}{7}$. The only critical point is $\frac{1}{7}(2,4,1)$.

Notice that $H_g = \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}$ and $Hf = \begin{pmatrix} 4&0&0\\0&2&0\\0&0&8 \end{pmatrix}$

which is positive-definite.

Since $\mathcal{W}\left(\frac{1}{7}(2,4,1), \frac{8}{7}\right) = Hf$ it follows that $f\left(\frac{1}{7}(2,4,1)\right)$ is a local min.

12. $f(x,y,z) = x+y+z$; $z = x^2+y^2$

$\nabla f = (1,1,1) \qquad \nabla g = (2x, 2y, -1)$

$1 = 2\lambda x \qquad (1)$

$1 = 2\lambda y \qquad (2)$

$1 = -\lambda \qquad (3)$

$z = x^2+y^2 \qquad (4)$

Equation (3) implies that $\lambda = -1$. Thus, from equations (1)-(3) $x = -\frac{1}{2}$, $y = -\frac{1}{2}$. Plugging these values into (4), we get $z = (-\frac{1}{2})^2 + (-\frac{1}{2})^2 = \frac{1}{2}$

Thus $\frac{1}{2}(-1,-1,1)$ is a critical point corresponding to $\lambda = -1$.

Notice that $Hf = \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}$ and $H_g = \begin{pmatrix} 2&0&0\\0&2&0\\0&0&0 \end{pmatrix}$.

Testing $Hf - (-1)H_g = H_g$ yields no information. However setting $f|_{z=x^2+y^2} = x+y+x^2+y^2 = (x+\frac{1}{2})^2 + (y+\frac{1}{2})^2 - \frac{1}{2}$ shows that $\frac{1}{2}(-1,-1,1)$ is a local min. Moreover, it is the absolute minimum on the constraint.

13. $f(x,y,z) = 3x^2+y^2+3z^2$; $x^2+y^2+z^2=1$; $x-y+5z=0$.

$$\nabla f = (6x, 2y, 6z) \quad \nabla g_1 = (2x, 2y, 2z)$$

$$\nabla g_2 = (1,-1,5)$$

We must solve the system of equations

$$6x = 2\lambda x + \mu \qquad (1)$$

$$2y = 2\lambda y - \mu \qquad (2)$$

$$6z = 2\lambda z + 5\mu \qquad (3)$$

$$x^2+y^2+z^2 = 1 \qquad (4)$$

$$x-y+5z = 0 \qquad (5)$$

We can rewrite equations (1) - (3) into the equivalent form

$$(6-2\lambda)x = \mu \qquad (6)$$

$$(2-2\lambda)y = -\mu \qquad (7)$$

$$(6-2\lambda)z = 5\mu \qquad (8)$$

If $\lambda=1$, then (7) implies that $\mu=0$. Substituting $\lambda=1, \mu=0$ into (6) and (8) implies that $x=z=0$. To satisfy (4), $y$ must be $\pm1$. This, however, does not satisfy (5). Hence $\lambda \neq 1$.

If $\lambda=3$ then $\mu=0$ by (6). Substituting $\lambda=3$ & $\mu=0$ in (7) implies that $y=0$. Substituting $y=0$ into (4) and (5), yields the equations

$$x^2 + z^2 = 1 \qquad (9)$$

$$x + 5z = 0 \qquad (10)$$

Thus $x = -5z$. Plugging in (9) gives $26z^2 = 1$ or $z = \frac{\pm 1}{\sqrt{26}}$.

Thus $\frac{1}{\sqrt{26}}(-5, 0, 1)$ and $\frac{1}{\sqrt{26}}(5, 0, -1)$ are critical points corresponding to $\lambda = 3$ and $\mu = 0$.

We can search for other critical points by "upgrading" equation (7). Add $4y$ to both sides of the equation to obtain

$$(6 - 2\lambda)y = -\mu + 4y \qquad (11)$$

Multiplying equation (6) by 1, (11) by $-1$, and (8) by 5 and adding, we obtain

$$(6 - 2\lambda)(x - y + 5z) = 27\mu - 4y \qquad (12)$$

And since $x - y + 5z = 0$ by equation (5), we see that $0 = 27\mu - 4y$ or

$$y = \frac{27}{4}\mu \qquad (13)$$

Since the case $\mu = 0$ has been considered, assume $\mu \neq 0$. Substituting (13) in (7) yields

$$(2 - 2\lambda)\frac{27}{4}\mu = -\mu.$$

Dividing by $\mu$ yields

$$(1-\lambda)\,\frac{27}{2} = -1 \quad \text{or}$$

$$\left(\frac{27}{2}+1\right)\frac{2}{27} = \lambda$$

Hence $\lambda = \frac{29}{27}$.

Notice now that, since $\mu \neq 0$ equations (6) and (8) imply that $5x = z$. Substituting into (4) and (5) we obtain

$$26x^2 + y^2 = 1 \qquad (14)$$

$$26x - y = 0 \qquad (15)$$

Thus $y = 26x$. Plugging into (14) we get that

$$26x^2 + 676x^2 = 1 \quad \text{or} \quad x = \frac{\pm 1}{\sqrt{702}}.$$

we obtain the critical points

$$\frac{1}{\sqrt{702}}(1, 26, 5) \text{ and } \frac{1}{\sqrt{702}}(-1, -26, -5)$$

which correspond to $\lambda = \frac{29}{27}$ and $\mu = \pm\frac{26 \cdot 4}{27\sqrt{702}}$.

Now $f\left(\pm\frac{1}{\sqrt{26}}(5, 0, -1)\right) = \frac{78}{26} \approx 3$

$$f\left(\frac{\pm 1}{\sqrt{702}}(1, 26, 5)\right) = \frac{754}{702} \approx 1.07$$

Hence, it appears that the relative max is 3 and relative min is $\frac{754}{702}$.

14. $f(x,y,z) = x+y+z$ ; $x^2+y^2+z^2 = 1$ ;

$x+2y+z=0$

$$\nabla f = (1,1,1) \qquad \nabla g_1 = (2x, 2y, 2z)$$

$$\nabla g_2 = (1,2,1)$$

We must solve the system of equations

$$1 = 2\lambda x + \mu \qquad (1)$$

$$1 = 2\lambda y + 2\mu \qquad (2)$$

$$1 = 2\lambda z + \mu \qquad (3)$$

$$x^2+y^2+z^2 = 1 \qquad (4)$$

$$x+2y+z = 0 \qquad (5)$$

Multiplying (1) by 1, (2) by 2, (3) by 1, and adding we get

$$1+2+1 = 2\lambda(x+2y+z) + \mu + 4\mu + \mu$$

By equation (5), this reduces to $4 = 6\mu$. Hence $\mu = \frac{2}{3}$.

Using $\mu = \frac{2}{3}$, rewrite equations (1)-(3) as

$$\frac{1}{3} = 2\lambda x \qquad (6)$$

$$-\frac{1}{3} = 2\lambda y \qquad (7)$$

$$\frac{1}{3} = 2\lambda z \qquad (8)$$

Squaring (6) − (7) and adding yields

$$3 \cdot \frac{1}{9} = 4\lambda^2 \left(x^2+y^2+z^2\right), \text{ which reduces}$$

to

$$\frac{1}{3} = 4\lambda^2 \qquad (9)$$

by equation (4).

Thus $\lambda = \pm \frac{1}{2\sqrt{3}}$

Setting $\lambda = \frac{1}{2\sqrt{3}}$ in equations (6)−(8) we obtain the critical point $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}(1,-1,1)$.

Setting $\lambda = \frac{-1}{2\sqrt{3}}$ in equations (6) − (8) we obtain the critical point $\frac{-1}{\sqrt{3}}(1,-1,1)$

Notice that $Hf = 0$, $Hg_1 = 2I_3$, and $Hg_2 = 0$.

Thus $W(\vec{a}, \lambda, \mu) = 2\lambda I_3$, is positive definite when $\lambda > 0$ and negative definite when $\lambda < 0$.

We therefore have a local min at $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and a local max at $\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$.

15. We can solve this problem by either applying the direct method of parametrizing the constraint or by the method of lagrange multipliers. We will take the latter approach.

$\nabla C = (10x+2y,\ 2x+6y)$, $\nabla g = (1,1)$

$$10x + 2y = \lambda \qquad (1)$$

$$2x + 6y = \lambda \qquad (2)$$

$$x + y = 39 \qquad (3)$$

Equations (1) & (2) imply that

$$2x+6y = 10x+2y$$

or $y = 2x \qquad (4)$

Substituting $2x$ into $y$ in equation (3) yields $3x = 39$ or $x = 13$. Hence, by equation (4), $y = 26$

setting $x=13$, $y=26$ in equation (2), for instance gives us $\lambda = 130 + 52 = 182$.

Thus our only critical point is $(13, 26)$ and it corresponds to $\lambda = 182$.

Now $H_g = 0$ and $Hf = \begin{pmatrix} 10 & 2 \\ 2 & 6 \end{pmatrix}$ which is positive definite. Thus $\mathcal{W}((13,26), 182) = Hf$ is positive definite, implying that the cost $C$ is minimized at $(13, 26)$ with the corresponding total cost of $C(13, 26) = 4349$ thousand dollars.

16.

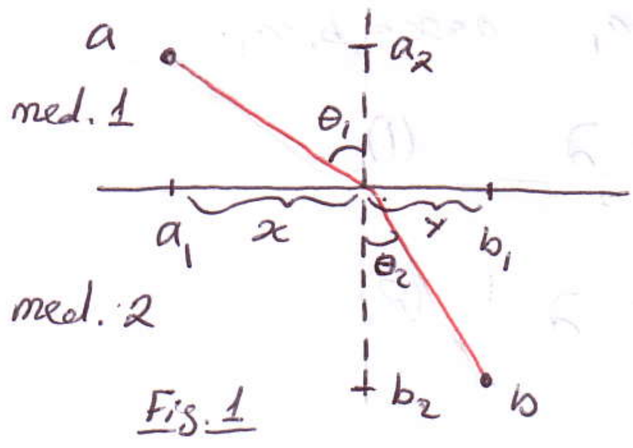


Fig. 1

The light will take the path from point $a=(a_1,a_2)$ to point $b=(b_1,b_2)$ that minimizes its time of travel. Suppose that the light arrives at the border between medium 1 and medium 2 $x$ units from $a_1$. Then it must travel the distance $D_1=\sqrt{a_2^2+x^2}$ in medium 1. To arrive at point b, the light must travel an additional distance $D_2 = \sqrt{b_2^2+y^2}$. Since the velocity of light in med. 1 is $v_1$ and the velocity in med. 2 is $v_2$, the total time of travel T is given by

$$T(x,y) = \frac{D_1}{v_1}+\frac{D_2}{v_2} = \frac{\sqrt{a_2^2+x^2}}{v_1}+\frac{\sqrt{b_2^2+y^2}}{v_2}$$

where $x$ and $y$ are subject to the constraint

$x+y=b_1-a_1, \quad 0\le x\le b_1-a_1$

Notice that the solution $x=0$ makes no physical sence since that woud imply that the light does not enter med. 2 and does not arrive in point b. By similar reasoning $y\neq 0$. Thus we must solve the equations

$$\nabla T = \lambda \nabla g$$

$$x+y=b_1-a_1, \quad 0<x<b_1-a_1$$

or

$$\frac{x}{v_1\sqrt{a_2^2+x^2}} = \lambda \qquad (1)$$

$$\frac{y}{v_2\sqrt{b_2^2+y^2}} = \lambda \qquad (2)$$

$$x+y=b_1-a_1 \; ; \quad 0<x<b_1-a_1 \qquad (3)$$

We can write (1) and (2) in terms of the angles $\theta_1$ and $\theta_2$ (see Fig. 1).

Notice that $\sin\theta_1 = \frac{x}{\sqrt{a_2^2+x^2}}$ and $\sin\theta_2 = \frac{y}{\sqrt{b_2^2+y^2}}$.

Since $x \neq 0$ and $y \neq 0$ we have

$$\frac{\sin\theta_1}{v_1} = \lambda = \frac{\sin\theta_2}{v_2} \quad \text{where } \lambda \neq 0$$

Thus $\frac{v_2}{v_1} = \frac{\sin\theta_2}{\sin\theta_1}$. To see that this requirement corresponds to a minimum value of $T$, observe that if $(x,y)$ and $\lambda$ satisfy equations (1)-(3), then

$$\omega((x,y),\lambda) = Hf(x,y) = \begin{pmatrix} a_2^2/(a_2^2+x^2)^{3/2} & 0 \\ 0 & b_2^2/(b_2^2+y^2)^{3/2} \end{pmatrix}$$

which is positive definite.

17. a) The Lagrange multiplier equations are

$$r_1 - 2as_1^2\omega_1 = \lambda \qquad (1)$$

$$r_2 - 2as_2^2\omega_2 = \lambda \qquad (2)$$

$$r_3 - 2as_3^2\omega_3 = \lambda \qquad (3)$$

$$r_4 - 2as_4^2\omega_4 = \lambda \qquad (4)$$

$$\omega_1+\omega_2+\omega_3+\omega_4 = 1 \qquad (5)$$

setting (1) = (2), (1) = (3), and (1) = (4) we obtain

$$2a(s_2^2\omega_2 - s_1^2\omega_1) = r_2 - r_1$$

$$2a(s_3^2\omega_3 - s_1^2\omega_1) = r_3 - r_1$$

$$2a(s_4^2\omega_4 - s_1^2\omega) = r_4 - r_1$$

so

$$\omega_2 = \frac{r_2 - r_1}{2s_2^2 a} + \frac{s_1^2}{s_2^2}\omega_1$$

$$\omega_3 = \frac{r_3 - r_1}{2s_3^2 a} + \frac{s_1^2}{s_3^2}\omega_1$$

$$\omega_4 = \frac{r_4 - r_1}{2s_4^2 a} + \frac{s_1^2}{s_4^2}\omega_1$$

Putting these values in equation (5) and solving for $\omega_1$ gives

$$\omega_1 = \frac{1 - \frac{r_2 - r_1}{2as_2^2} - \frac{r_3 - r_1}{2as_3^2} - \frac{r_4 - r_1}{2as_4^2}}{1 + \frac{s_1^2}{s_2^2} + \frac{s_1^2}{s_2^2} + \frac{s_1^2}{s_4^2}}$$

The values of $\omega_2$, $\omega_3$, and $\omega_4$ then follow immediately. To see that our solution to (1) - (5) is a maximum, observe that $\mathcal{W}((\omega_1, \omega_2, \omega_3, \omega_4), \lambda) = Hf(\omega_1, \omega_2, \omega_3, \omega_4) =$

$$= \begin{pmatrix} -2as_1^2 & 0 & 0 & 0 \\ 0 & -2as_2^2 & 0 & 0 \\ 0 & 0 & -2as_3^2 & 0 \\ 0 & 0 & 0 & -2as_4^2 \end{pmatrix}$$ is negative definite.

b) $a = \dfrac{0.08 - 0.05}{0.04^2 - 0.01^2} = 20$