Solutions to H.W # 17

1. The surfaces $z = x^2+y^2$ and $z = 1-x^2-y^2$ intersect when $x^2+y^2 = 1-x^2-y^2$. In other words, when $2(x^2+y^2)=1$. The latter equation reduces to $\sqrt{x^2+y^2} = \frac{1}{\sqrt{2}}$.

Since $z = x^2+y^2$ and $z = 1-x^2-y^2$ are surfaces of revolution, they are simply described in cylindrical coordinates as $w = z = r^2$ and $w = 1-r^2$. The solid of integration S is therefore given by

$$0 \le r \le \frac{1}{\sqrt{2}}$$
$$0 \le \theta \le 2\pi$$
$$r^2 \le w \le 1-r^2$$

Thus,

$$\iiint_S \left(1+\frac{x}{\sqrt{x^2+y^2}}\right) = \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\int_{r^2}^{1-r^2}\left(1+\frac{r\cos\theta}{r}\right) r\,dw\,dr\,d\theta =$$

$$= \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\int_{r^2}^{1-r^2}(r+r\cos\theta)\,dw\,dr\,d\theta = \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}(1-2r^2)(r+r\cos\theta)\,dr$$

$$= \left(\int_0^{2\pi}(1+\cos\theta)\,d\theta\right)\left(\int_0^{\frac{1}{\sqrt{2}}}(1-2r^2)r\,dr\right) =$$

$$= 2\pi\int_0^{\frac{1}{\sqrt{2}}}(r-2r^3)\,dr = 2\pi\left(\frac{r^2}{2}-\frac{r^4}{2}\right)\Bigg|_0^{\frac{1}{\sqrt{2}}} = \pi\left(\frac{1}{2}-\frac{1}{4}\right) =$$

$$= \frac{\pi}{4}.$$

2. The surfaces $z=\sqrt{x^2+y^2}$ and $z=1-x^2-y^2$ intersect when $\sqrt{x^2+y^2}=1-(x^2+y^2)$ or, in cylindrical coordinates, wherever $r=1-r^2$. The latter equation reduces to the quadratic equation $r^2+r-1=0$, $r\geq 0$ or $r=\frac{-1+\sqrt{5}}{2}$

Notice that S can be described in cylindrical coordinates by

$$0\leq r\leq \frac{-1+\sqrt{5}}{2}$$
$$0\leq \theta\leq 2\pi$$
$$r\leq w\leq 1-r^2$$

Thus,

$$\iiint_S z = \int_0^{2\pi}\int_0^{\frac{-1+\sqrt{5}}{2}}\int_r^{1-r^2} rw\,dw\,dr\,d\theta =$$

$$= 2\pi\int_0^{\frac{-1+\sqrt{5}}{2}} r\left(\frac{(1-r^2)^2}{2}-\frac{r^2}{2}\right)dr = 2\left(\frac{-3}{16}+\frac{5\sqrt{5}}{48}\right)\pi.$$

3. The region S is bounded below by the plane $z=-1$ and above by the paraboloid $z=3-x^2-y^2$. The solid S is carved by the half-cylinder x^2+y^2, $x\leq 0$. S can therefore be described in cylindrical coordinates as

$$0\leq r\leq 1$$
$$\frac{\pi}{2}\leq\theta\leq\frac{3\pi}{2}$$
$$-1\leq w\leq 3-r^2$$

Thus,

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-1}^{3-x^2-y^2} z\,dz\,dy\,dx = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_0^1\int_{-1}^{3-r^2} rw\,dw\,dr\,d\theta =$$

$$= \frac{\pi}{2}\int_0^1 r\left(\frac{(3-r^2)^2}{2}-\frac{1}{2}\right)dr = \frac{2\pi}{3}$$

4. The region S is bounded below by the paraboloid $z=x^2+y^2$ and above by the plane $z=4$. It is carved out by the cylinder $x^2+(y-\frac{1}{2})^2=\frac{1}{4}$. Letting $r=\sqrt{x^2+y^2}$, $x=r\cos\theta$, and $y=r\sin\theta$, we see that S is described in cylindrical coordinates as

$$0\le r\le \sin\theta$$
$$0\le\theta\le\pi$$
$$r^2\le w\le 4$$

Thus,

$$\int_0^1\int_{-\sqrt{\frac{1}{4}-(y-\frac{1}{2})^2}}^{\sqrt{\frac{1}{4}-(y-\frac{1}{2})^2}}\int_{x^2+y^2}^{4}(x+y)\,dz\,dx\,dy =$$

$$= \int_0^{\pi}\int_0^{\sin\theta}\int_{r^2}^{4} r(r\cos\theta+r\sin\theta)\,dw\,dr\,d\theta =$$

$$= \int_0^{\pi}\int_0^{\sin\theta}(4r^2-r^4)(\cos\theta+\sin\theta)\,dr\,d\theta =$$

$$= \int_0^{\pi}\left(\frac{4}{3}\sin^3\theta-\frac{1}{5}\sin^5\theta\right)(\cos\theta+\sin\theta)\,d\theta = \frac{7\pi}{16}$$

5. The solid S is bounded below by the paraboloid x^2+y^2-1, above by the cone $\sqrt{x^2+y^2}$, and is carved by the eighth cylinder $x^2+y^2=1$, $0 \le x \le \frac{1}{\sqrt{2}}$. Therefore S is described in cylindrical coordinates by

$$0 \le r \le 1$$
$$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$$
$$r^2-1 \le w \le r$$

Thus,

$$\int_0^{1/\sqrt{2}} \int_x^{\sqrt{1-x^2}} \int_{x^2+y^2-1}^{\sqrt{x^2+y^2}} (x^2+y^2)\, dz\, dy\, dx =$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 \int_{r^2-1}^{r} r(r^2)\, dw\, dr\, d\theta = \frac{\pi}{4} \int_0^1 (r-r^2+1) r^3\, dr =$$

$$= \frac{\pi}{4} \int_0^1 (r^4-r^5+r^3)\, dr = \frac{17\pi}{240}$$

6. The solid is simply described in spherical coordinates as

$$0 \le \rho \le 3$$
$$0 \le \theta \le \frac{\pi}{2}$$
$$0 \le \varphi \le \frac{\pi}{2}$$

Thus,

$$\iiint_S \sqrt{x^2+y^2+z^2} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho^3 \sin\varphi\, d\rho\, d\varphi\, d\theta =$$

$$= \left(\int_0^{\frac{\pi}{2}} d\theta\right)\left(\int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi\right)\left(\int_0^3 \rho^3 d\rho\right) = \frac{81\pi}{8}$$

7. This solid is simply described in spherical coordinates as

$$0 \le \rho \le 4$$
$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le \frac{\pi}{6} \quad (\text{Why?})$$

Thus,

$$\iiint_S (x+y) = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^4 (\rho\sin\varphi\cos\theta + \rho\sin\varphi\sin\theta)\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^4 \rho^3\sin^2\varphi(\cos\theta+\sin\theta)\, d\rho\, d\varphi\, d\theta =$$

$$= \left(\int_0^{2\pi}(\cos\theta+\sin\theta)\,d\theta\right)\left(\int_0^{\frac{\pi}{6}}\sin^2\varphi\, d\varphi\right)\left(\int_0^4 \rho^3 d\rho\right) = 0$$

because $\int_0^{2\pi}(\cos\theta+\sin\theta)\,d\theta = 0$ (How can this be established without computing the integral?)

8. The solid S is bounded below by $z=0$ and above by $z=\sqrt{x^2+y^2}$. Hence $\frac{\pi}{4} \le \varphi \le \frac{\pi}{2}$. S is bounded on the sides by the planes $y=\sqrt{3}x$ and $y=x$, so $\frac{\pi}{4} \le \theta \le \frac{\pi}{3}$. Finally, S is wedged between the spheres $x^2+y^2+z^2=2$ and $x^2+y^2+z^2=8$, so $\sqrt{2} \le \rho \le \sqrt{8}$.

Thus,

$$\iiint_S \frac{x}{\sqrt{x^2+y^2+z^2}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_{\sqrt{2}}^{\sqrt{8}} \frac{(\rho \sin\varphi)^2 \cos\theta}{\rho} \, d\rho \, d\theta \, d\varphi =$$

$$= \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2\varphi \, d\varphi\right)\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos\theta \, d\theta\right)\left(\int_{\sqrt{2}}^{\sqrt{8}} \rho \, d\rho\right) =$$

$$= \frac{3}{16}\left(\sqrt{3}-\sqrt{2}\right)(2+\pi)$$

9. The region of integration S is the ball $x^2+y^2+z^2 \leq 4$. Therefore,

$$\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{-\sqrt{4-y^2-z^2}}^{\sqrt{4-y^2-z^2}} (x^2+y^2+z^2)^{3/2} \, dx \, dz \, dy =$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^2 \rho^3(\rho^2 \sin\varphi) \, d\rho \, d\varphi \, d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\varphi \, d\varphi\right)\left(\int_0^2 \rho^5 \, d\rho\right)$$

$$= 2\pi \cdot 2 \cdot \frac{2^6}{6} = \frac{128\pi}{3}$$

10. The region S is bounded in the first octent by the planes $x=y$, $x=0$, the sphere $x^2+y^2+z^2=1$ and the cone $z=\sqrt{x^2+y^2}$.

Thus,

$$\int_0^{1/2} \int_y^{\sqrt{1/2 - y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} \frac{1}{1+(x^2+y^2+z^2)^{3/2}} \, dz \, dx \, dy =$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \int_0^1 \frac{\rho^2 \sin\varphi}{1+\rho^3} \, d\rho \, d\varphi \, d\theta = -\frac{1}{24}\left(-2+\sqrt{2}\right)\pi \ln 2$$

11. The region S is bounded in the first octant by the sphere $x^2+y^2+z^2=9$

Therefore,

$$\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}} xy\,dz\,dy\,dx = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^3 \rho^4 \sin^3\varphi\, d\rho\, d\varphi\, d\theta =$$

$$= \left(\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\, d\theta\right)\left(\int_0^{\frac{\pi}{2}} \sin^3\varphi\, d\varphi\right)\left(\int_0^3 \rho^4\, d\rho\right) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3^5}{5} =$$

$$= \frac{81}{5}$$

12. The region of integration is located in the first octant. The region is bounded by the planes $y=x$ and $x=0$ and contained inside the sphere $x^2+y^2+z^2=1$.

Thus,

$$\int_0^{\frac{1}{\sqrt{2}}}\int_y^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} (z+y)\,dz\,dx\,dy =$$

$$= \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^2\sin\varphi\left(\rho\cos\varphi + \rho\sin\varphi\sin\theta\right)d\rho\, d\varphi\, d\theta =$$

$$= \left(\int_0^1 \rho^3\, d\rho\right)\left(\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}} (\cos\varphi\sin\varphi + \sin^2\varphi\sin\theta)\, d\varphi\, d\theta\right) =$$

$$= -\frac{1}{32}\left(-3+\sqrt{2}\right)\pi$$

13. Let $s = x+y+3z$, $t = y+z$, $u = x+z$. Then the desired region is

$$0 \le s \le 1$$
$$0 \le t \le 1$$
$$-1 \le u \le 1$$

and

$$\det \frac{\partial(s,t,u)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1-(-1)-3 = 2-3 = -1$$

Hence $\left|\det \dfrac{\partial(x,y,z)}{\partial(s,t,u)}\right| = 1$.

Thus,

$$\iiint_S (x+z)^2 = \int_0^1\int_0^1\int_{-1}^1 u^2\,du\,dt\,ds = \int_{-1}^1 u^2\,du = 2\int_0^1 u^2\,du =$$
$$= \frac{2}{3}$$

14. Let $s = \frac{z}{x}$, $t = y+z$, $u = x$. Then the desired region is

$$\frac{1}{3} \le s \le 4$$
$$2 \le t \le 5$$
$$6 \le u \le 8$$

Now, $\det \dfrac{\partial(s,t,u)}{\partial(x,y,z)} = \begin{vmatrix} -\frac{z}{x^2} & 0 & \frac{1}{x} \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = -\dfrac{1}{x}$

Therefore,

$$\left| \det \frac{\partial(x,y,z)}{\partial(s,t,u)} \right| = x$$

Observe that $z = sx = su$

$$y = t - z = t - su$$

$$z^2 = (su)^2$$

so $yz + z^2 = (t-su)(su) + (su)^2 = stu$.

Thus,

$$\iiint_S \frac{yz+z^2}{x} = \int_{1/3}^{4}\int_{2}^{5}\int_{6}^{8} stu\,du\,dt\,ds = \frac{7007}{6}$$

15. Let $s = z - \sin y$, $t = y$, $u = x + z$. Then the region is simply described as

$$-1 \le s \le 1$$

$$0 \le t \le \frac{\pi}{2}$$

$$0 \le u \le 1$$

Now, $\det \dfrac{\partial(s,t,u)}{\partial(x,y,z)} = \begin{vmatrix} 0 & -\cos y & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -1$

Hence $\left| \det \dfrac{\partial(x,y,z)}{\partial(s,t,u)} \right| = 1$

Thus,

$$\iiint_S \cos y = \int_{-1}^{1}\int_0^{\frac{\pi}{2}}\int_0^1 \cos t \, du\, dt\, ds = 2\int_0^{\frac{\pi}{2}} \cos t\, dt =$$

$$= 2 \sin t \Big|_0^{\frac{\pi}{2}} = 2$$