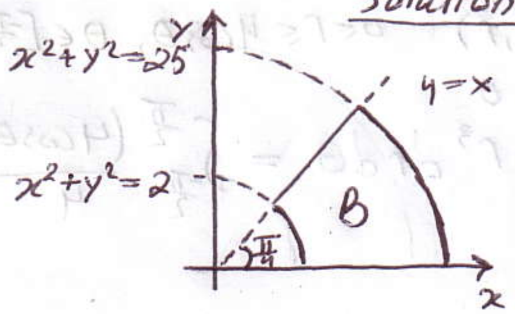
Solutions to H.W. #16

1.



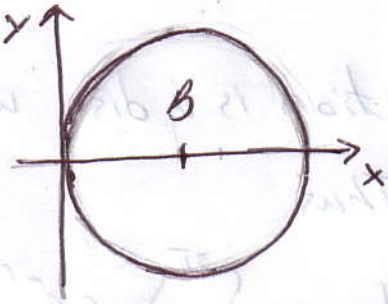
$$\iint_B \frac{y}{x} = \int_0^{\frac{\pi}{4}}\int_{\sqrt{2}}^{5} \frac{r\sin\theta}{r\cos\theta}\, r\,dr\,d\theta = \int_0^{\frac{\pi}{4}}\int_{\sqrt{2}}^{5} r\tan\theta\,dr\,d\theta =$$

$$= \left(\int_{\sqrt{2}}^{5} r\,dr\right)\left(\int_0^{\frac{\pi}{4}} \tan\theta\,d\theta\right) = \frac{23}{2}\ln\sqrt{2} = \frac{23}{4}\ln 2$$

2. $$\int_{\frac{\pi}{2}}^{\pi}\int_0^1 r^4\,dr\,d\theta = \frac{\pi}{2}\cdot\frac{1}{5} = \frac{\pi}{10}$$

3. $$\iint_B \frac{1}{\sqrt{x^2+y^2}} = \int_0^{2\pi}\int_2^3 \frac{1}{r}\, r\,dr\,d\theta = 2\pi$$

4. $x^2+y^2 \leq 4x \Rightarrow x^2-4x+4+y^2 \leq 4$ or $(x-2)^2+y^2 \leq 4$. In particular, B is the disc whose center is $(2,0)$ and whose radius is also 2.



Let $f: [-\pi,\pi]\times[0,\infty) \to \mathbb{R}^2$ be given by $f(\theta,r) = (r\cos\theta, r\sin\theta)$. Then $|\det Jf(\theta,r)| = r$

and $B = f(C)$ where $C = \{(\theta, r) : 0 \le r \le 4\cos\theta, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$.

Thus $\iint_B (x^2+y^2) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{4\cos\theta} r^3 \, dr \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(4\cos\theta)^4}{4} d\theta$

$= 24\pi$

5. The lines $y = x/\sqrt{3}$ and $y = x$ are the same as $r\sin\theta = r\cos\theta/\sqrt{3}$ and $r\sin\theta = r\cos\theta$ in polar coordinates. This simplifies to $\tan\theta = \frac{1}{\sqrt{3}}$ and $\tan\theta = 1$. Hence

$$\iint_B \tan^{-1}\left(\frac{y}{x}\right) = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_{\frac{1}{2}}^{1} r \tan^{-1}(\tan\theta) \, dr \, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_{\frac{1}{2}}^{1} r\theta \, dr \, d\theta$$

$$= \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \theta \, d\theta\right)\left(\int_{\frac{1}{2}}^{1} r \, dr\right) = \frac{\left(\left(\frac{\pi}{4}\right)^2 - \left(\frac{\pi}{6}\right)^2\right)\left(1 - \frac{1}{4}\right)}{4}$$

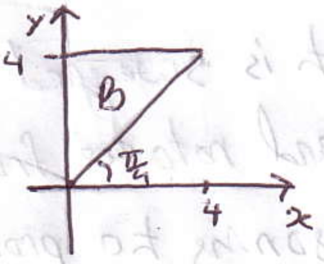
6. The region of integration is a circular sector ($\frac{1}{4}$ circle). Therefore,

$$\int_0^1 \int_0^{\sqrt{1-x^2}} x^2 y \, dy \, dx = \int_0^{\frac{\pi}{2}} \int_0^1 r^4 \cos^2\theta \sin\theta \, dr \, d\theta =$$

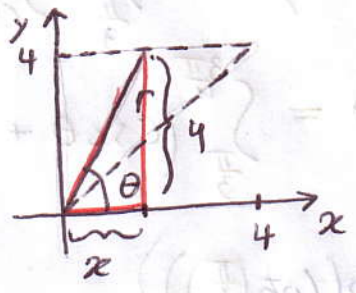
$$= \frac{1}{5} \int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta \, d\theta = -\frac{1}{5}\left(\frac{1}{3}\cos^3\theta \Big|_0^{\frac{\pi}{2}}\right) = \frac{1}{15}.$$

7. The region of integration is a disc with center at the origin and radius 2. Thus

$$\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-x^2-y^2} \, dx \, dy = \int_0^{2\pi} \int_0^2 r e^{-r^2} \, dr \, d\theta = \pi - \frac{\pi}{e^4}$$

8. 

The region B is a triangle. To describe it in polar coordinates notice that as θ varies from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, r must extend from 0 to the line $y=4$. Therefore $r \leq \sqrt{x^2+16}$ where $\frac{x}{4} = \cot\theta$ as can be verified from the picture below.



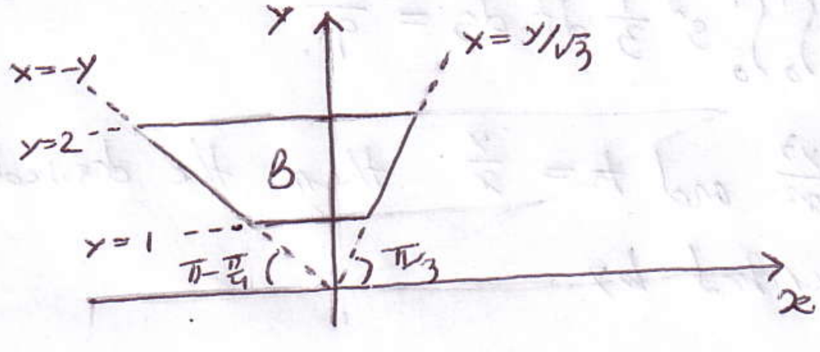
$\frac{4}{x} = \tan\theta$ so $\frac{x}{4} = \cot\theta$

Therefore, $0 \leq r \leq \sqrt{16\cot^2\theta + 16} = 4|\csc\theta| = 4\csc\theta$

In particular,

$$\int_0^4 \int_x^4 (x^2+y^2)^{3/2}\, dy\, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{4\csc\theta} r^4\, dr\, d\theta =$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{(4\csc\theta)^5}{5}\, d\theta = \frac{128}{5}\left(7\sqrt{2} - 3\ln\left(\tan\left(\frac{\pi}{8}\right)\right)\right)$$

9. 

To describe region B, observe that it is swepped by a line segment that varies in length and rotates from $\theta = \frac{\pi}{3}$ to $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. By similer reasoning to problem 8,

$$\sqrt{x^2+1} \leq r \leq \sqrt{x^2+4}, \text{ or}$$

$$\csc\theta \leq r \leq 2\csc\theta$$

Thus,

$$\int_1^2 \int_{-y}^{y/\sqrt{3}} \frac{1+\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,dx\,dy = \int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} \int_{\csc\theta}^{2\csc\theta} (1+r)\,dr\,d\theta =$$

$$= \int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} \left(r + \frac{r^2}{2}\right)\Big|_{\csc\theta}^{2\csc\theta} d\theta = \int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} \left(\csc\theta + \frac{3\csc^2\theta}{2}\right) d\theta =$$

$$= \frac{1}{2}\left(3+\sqrt{3} + \ln 3 + 2\ln\left(\cot\left(\tfrac{\pi}{8}\right)\right)\right)$$

10. Let $s = x+y$ and $t = 2x-y$. With this substitution

$$0 \leq s \leq 1$$
$$0 \leq t \leq 1$$

Now $\left|\det \frac{\partial(s,t)}{\partial(x,y)}\right| = \left|\det\begin{pmatrix} 1 & 1 \\ 2 & -1\end{pmatrix}\right| = |-3| = 3.$

Thus $\left|\det \frac{\partial(x,y)}{\partial(s,t)}\right| = \frac{1}{3}$

It follows that

$$\iint_B (x+y)^2 = \int_0^1\int_0^1 s^2 \frac{1}{3}\,dt\,ds = \frac{1}{9}.$$

11. Let $s = \frac{y^3}{x^2}$ and $t = \frac{y}{x}$ then the desired region would be described by

$$1 \le s \le 6$$
$$2 \le t \le 3$$

Now,
$$\det \frac{\partial(s,t)}{\partial(x,y)} = \det \begin{pmatrix} -2\frac{y^3}{x^3} & 3\frac{y^2}{x^2} \\ \frac{y}{x^2} & \frac{1}{x} \end{pmatrix} =$$

$$= \frac{y^3}{x^4}$$

Thus, $\det \frac{\partial(x,y)}{\partial(s,t)} = \frac{x^4}{y^3}$ and

$$\iint_B \frac{1}{y} = \int_1^6 \int_2^3 \frac{1}{y(s,t)} \cdot \frac{x^4(s,t)}{y^3(s,t)}\, dt\, ds = \int_1^6 \int_2^3 \frac{1}{\left(\frac{y}{x}\right)^4}\, dt\, ds =$$

$$= \int_1^6 \int_2^3 t^{-4}\, dt\, ds = -\frac{5}{3} t^{-3}\Big|_2^3 = \frac{5}{3} 2^{-3} - \frac{5}{3} 3^{-3}$$

12. let $s = \frac{x^2}{y}$ and $t = \frac{y^2}{x}$. Then

$$\frac{\pi}{2} \le s \le \pi$$
$$2 \le t \le 3$$

Notice that $xy = \frac{x^2}{y} \cdot \frac{y^2}{x} = st$ and $\frac{x^2}{y} = s$.

Hence $\frac{x^2 \sin xy}{y} = s \sin st$.

Now $\det \frac{\partial(s,t)}{\partial(x,y)} = \det \begin{pmatrix} 2\frac{x}{y} & -\frac{x^2}{y^2} \\ \frac{y^2}{x^2} & 2\frac{y}{x} \end{pmatrix} = 3$

Therefore $\det \frac{\partial(x,y)}{\partial(s,t)} = \frac{1}{3}$. In particular,

$$\iint_B \frac{x^2 \sin xy}{y} = \int_{\frac{\pi}{2}}^{\pi} \int_2^3 \frac{1}{3} s \sin st.\, dt\, ds = -\frac{1}{9}$$

13. Let $s = \frac{y}{x}$ and $t = x+y$. then

$$1 \le s \le 3$$
$$-3 \le t \le -1$$

and $\det \frac{\partial(s,t)}{\partial(x,y)} = \det \begin{pmatrix} \frac{-y}{x^2} & \frac{1}{x} \\ 1 & 1 \end{pmatrix} = \frac{-(x+y)}{x^2}$

Since $x^2 > 0$ and $t = x+y < 0$ (why?),

$$\left| \det \frac{\partial(s,t)}{\partial(x,y)} \right| = \frac{-(x+y)}{x^2}$$

Hence $\left| \det \frac{\partial(x,y)}{\partial(s,t)} \right| = \frac{-x^2}{x+y}$.

Therefore

$$\iint_B x^2 = \int_1^3 \int_{-3}^{-1} x^2 \frac{-x^2}{x+y}\, dt\, ds = \int_1^3 \int_{-3}^{-1} \frac{-x^4}{x+y}\, dt\, ds =$$

$$= \int_1^3 \int_{-3}^{-1} -\frac{x^4}{t}\, dt\, ds \quad (1)$$

To complete (1), observe that $s = \frac{y}{x}$ implies that $x = \frac{y}{s}$. Now, $t = x+y = \frac{y}{s} + y = y\left(\frac{s+1}{s}\right)$. Hence, $y = \frac{st}{s+1}$ and $x = \frac{st}{s+1} / s = \frac{t}{s+1}$ so $x^4 = \frac{t^4}{(s+1)^4}$.
It follows that $(1) = \int_1^3 \int_{-3}^{-1} -\frac{t^3}{(s+1)^4}\, dt\, ds =$

$$= \left(\int_1^3 \frac{1}{(s+1)^4}\, ds \right) \left(\int_{-3}^{-1} -t^3\, dt \right) = \frac{7}{192} \cdot 20 = \frac{35}{48}$$

14. Let $s = \frac{x^2}{4} + \frac{y^2}{9}$ and $t = \frac{y}{x^2}$

then

$$1 \le s \le 4$$
$$\tfrac{1}{2} \le t \le 2.$$

Now $\det \frac{\partial(s,t)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{2} & \frac{2}{9}y \\ -2\frac{y}{x^3} & \frac{1}{x^2} \end{vmatrix} = \frac{1}{2x} + \frac{4y^2}{x^3} =$

$$= \frac{9x^2+8y^2}{18x^3}$$

Thus,

$$\det \frac{\partial(x,y)}{\partial(s,t)} = \frac{18x^3}{9x^2+8y^2}$$

Therefore,

$$\iint_B \frac{9x^2+8y^2}{xy} = \int_1^4 \int_{1/2}^2 \frac{9x^2+8y^2}{xy} \cdot \left| \frac{18x^3}{9x^2+8y^2} \right| dt\,ds =$$

$$= \int_1^4 \int_{1/2}^2 18 \frac{|x|}{x} \frac{x^2}{y} \, dt\,ds \quad (1).$$

Since B is in the first quadrant, $x > 0$. Hence

$$(1) = \int_1^4 \int_{1/2}^2 18 \frac{x^2}{y} \, dt\,ds = \int_1^4 \int_{1/2}^2 18 \frac{1}{t} \, dt\,ds =$$

$$= 18 \cdot 3 \cdot \left(\ln 2 - \ln \tfrac{1}{2} \right) = 18 \cdot 3 \left(2 \ln 2 \right) = 108 \ln 2.$$

15. $\iint_{\mathbb{R}^2} e^{-x^2-y^2} = \lim_{c\to\infty} \iint_{D_c} e^{-x^2-y^2}$ where

D_c is the disc with center $(0,0)$ and radius c.

Now $\lim_{c\to\infty} \iint_{D_c} e^{-x^2-y^2} = \lim_{c\to\infty} \int_0^{2\pi}\int_0^c re^{-r^2}\,dr\,d\theta =$

$= 2\pi \lim_{c\to\infty} -\frac{1}{2}e^{-r^2}\Big|_0^c = -\pi \lim_{c\to\infty}(e^{-c^2}-1) = \pi$

16. $\iint_B \frac{1}{1+(x^2+y^2)^2} = \lim_{c\to\infty} \iint_{B\cap D_c} \frac{1}{1+(x^2+y^2)^2}$

where D_c is the same as in the previous problem.

The region $B\cap D_c$ can be simply described in polar coordinates as $0 \le \theta \le \frac{\pi}{4}$, $0 \le r \le c$.

Thus,

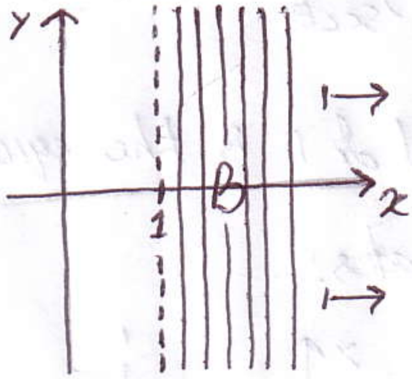
$\lim_{c\to\infty} \iint_{B\cap D_c} \frac{1}{1+(x^2+y^2)^2} = \lim_{c\to\infty} \int_0^{\frac{\pi}{4}}\int_0^c \frac{r}{1+(r^2)^2}\,dr\,d\theta =$

$= \frac{\pi}{4} \lim_{c\to\infty} \int_0^c \frac{r}{1+(r^2)^2}\,dr = \frac{\pi}{8} \lim_{c\to\infty} \int_0^c \frac{2r}{1+(r^2)^2}\,dr =$

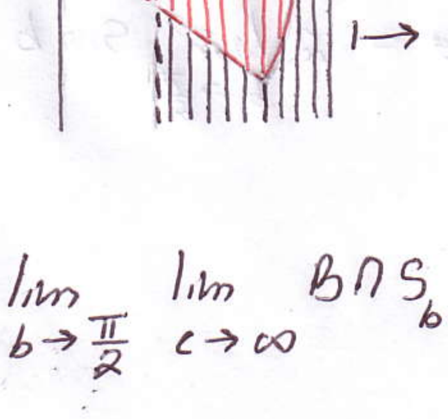
$= \frac{\pi}{8} \lim_{c\to\infty} \tan^{-1}(r^2)\Big|_0^c = \frac{\pi}{8} \lim_{c\to\infty} \tan^{-1}(c^2) = \frac{\pi}{8}\cdot\frac{\pi}{2} =$

$= \frac{\pi^2}{16} = \left(\frac{\pi}{4}\right)^2$

17. The region of integration B is the shaded region in the picture below.



Let $b \in (0, \frac{\pi}{2})$ and $c > 0$. Then $-b \leq \theta \leq b$, $0 \leq r \leq c$ describes a segment of a disc of radius c and angle $2b$: Let's agree to call this region S_{bc}. If $c > 1$, $B \cap S_{bc} \neq \emptyset$. The picture below displays this intersection with red ink.

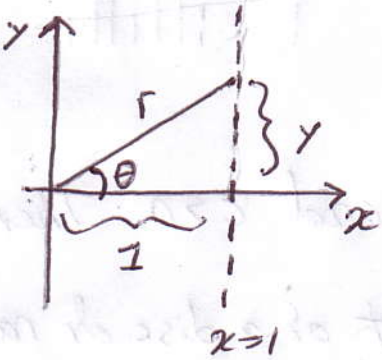


Notice that $B = \lim_{b \to \frac{\pi}{2}} \lim_{c \to \infty} B \cap S_{bc}$

Thus, $\iint\limits_{B} \frac{1}{(x^2+y^2)^{3/2}} = \lim\limits_{b\to\frac{\pi}{2}}\ \lim\limits_{c\to\infty} \int_{-b}^{b}\int_{\sec\theta}^{c} \frac{1}{r^3}\, r\, dr\, d\theta$

$= 2 \lim\limits_{b\to\frac{\pi}{2}}\ \lim\limits_{c\to\infty} \int_{0}^{b}\int_{\sec\theta}^{c} \frac{1}{r^2}\, dr\, d\theta$ (1)

where the lower bound of r is the equation of the line $x=1$ in polar coordinates:



$r = \sqrt{1+y^2}$; $y = \frac{y}{1} = \tan\theta$. Therefore $r = \sqrt{1+\tan^2\theta} =$
$= \sec\theta$

Therefore

$(1) = 2 \lim\limits_{b\to\frac{\pi}{2}}\ \lim\limits_{c\to\infty} \int_0^b -\frac{1}{r}\Big|_{\sec\theta}^{c} db = 2 \lim\limits_{b\to\frac{\pi}{2}} \int_0^b \left(\cos\theta - \frac{1}{c}\right) db \Big|_{c\to\infty} =$

$= 2 \lim\limits_{b\to\frac{\pi}{2}} \int_0^b \cos\theta\, d\theta = 2 \lim\limits_{b\to\frac{\pi}{2}} \sin b = 2 \sin\frac{\pi}{2} = 2$