Solutions to H.W #15

1. $\int_0^1\int_1^2\int_2^3 \frac{z e^x}{y}\,dz\,dy\,dx = \left(\int_0^1 e^x\,dx\right)\left(\int_1^2 \frac{1}{y}\,dy\right)\left(\int_2^3 z\,dz\right)$
$= (e-1)(\ln 2)\left(\frac{5}{2}\right).$

2. $\int_{-1}^1\int_{-1}^1\int_{x-1}^{y+1} xy\,dz\,dx\,dy = \int_{-1}^1\int_{-1}^1\left[(y+1)xy-(x-1)xy\right]dx\,dy$
$= \left(\int_{-1}^1 (y+1)y\,dy\right)\left(\int_{-1}^1 x\,dx\right) - \left(\int_{-1}^1 y\,dy\right)\left(\int_{-1}^1 (x-1)x\,dx\right) = 0$
because $\int_{-1}^1 x\,dx = \int_{-1}^1 y\,dy = 0$ (why?).

3. $\int_0^3\int_0^{6-2y}\int_0^{(6-2y-x)/3} y\,dz\,dx\,dy = \int_0^3\int_0^{6-2y} \frac{y(6-2y-x)}{3}\,dx\,dy$
$= \int_0^3 \frac{y(6-2y)^2}{6}\,dy = \frac{9}{2}$

4. $\int_0^1\int_0^1\int_0^{1+x^2} xyz\,dz\,dx\,dy = \int_0^1\int_0^1 xy\frac{(1+x^2)^2}{2}\,dx\,dy =$
$= \left(\int_0^1 y\,dy\right)\left(\int_0^1 x\frac{(1+x^2)^2}{2}\,dx\right) = \frac{1}{2}\cdot\frac{7}{12} = \frac{7}{24}.$

5. $\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} xy\,dz\,dy\,dx = \int_0^2\int_0^{\sqrt{4-x^2}} xy\sqrt{4-x^2-y^2}\,dy\,dx =$
$= \int_0^2 -\frac{1}{3}x(4-x^2-y^2)^{3/2}\Big|_0^{\sqrt{4-x^2}}dx = \int_0^2 \frac{1}{3}x(4-x^2)^{3/2}dx = \frac{32}{15}$

6. The boundaries are the planes $x=1$, $x=4$, $y=0$, $y=2$, $z=0$, and $z=5$. Thus $\iiint_S \sqrt{x+y+z} = \int_1^4\int_0^2\int_0^5 \sqrt{x+y+z}\,dz\,dy\,dx$

$= \frac{8}{105}\left(-2060 - 1024\sqrt{2} + 27\sqrt{3} + 1331\sqrt{11}\right)$

According to Mr. Computer.

7, Notise that both the solid and the function $f(x,y,z) = x$ that is to be integrated are symmetric about the yz-plane. In other words, $f(-x,y,z) = -f(x,y,z)$ and if $(x,y,z) \in S$ so does $(-x,y,z)$.

Therefore $\iiint_S f(x,y,z) = \iiint_S x = 0.$

Just for fun, let us still write it out as an iterated integral:

$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{x^2+y^2}^{1-(x^2+y^2)} x \, dz \, dy \, dx = 0$$

where the boundaries of integration are determined by setting $1-(x^2+y^2) = x^2+y^2$.

8, This solid is a tetrahedron. It is x-, y-, and z-simple.

Treating it as x-simple, we see that

$$S = \{(y,z) \in B : 0 \le x \le 8-2y-4z\}$$

where B is the 2-D region in the yz-plane obtained by projecting S onto the yz-plane (i.e. if $(x,y,z) \in S$, this point is projected onto $(0,y,z)$).

The boundaries of B are the lines $y=0$, $z=0$ and

$0=x=8-2y-4z$ or $y+2z=4$

Therefore,

$$B=\{(y,z): 0\le y\le 4-2z,\ 0\le z\le 2\}$$

Thus, the desired integral is

$$\int_0^2\int_0^{4-2z}\int_0^{8-2y-4z} xy\,dx\,dy\,dz = \frac{256}{15}$$

according to Mr. computer.

9. The setup of this integral is similar to the one in problem 7,

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} x^2\,dz\,dy\,dx =$$

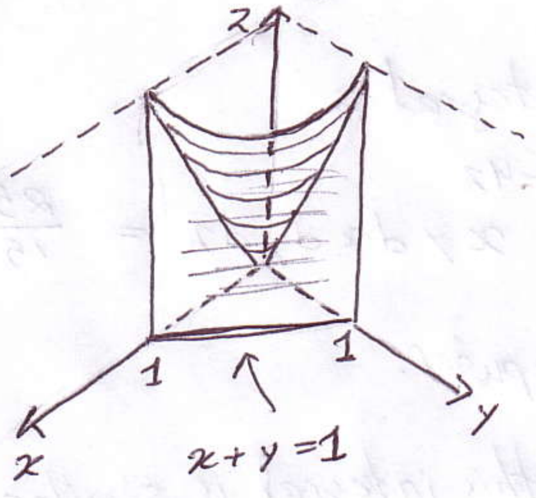
$$= \int_{-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(8x^2-2x^2(x^2+y^2)\right)dy\,dx =$$

$$= \int_{-2}^{2}\left(16x^2\sqrt{4-x^2} - 4x^4\sqrt{4-x^2} - \frac{4}{3}x^2(4-x^2)^{3/2}\right)dx$$

$= \frac{32\pi}{3}$ where we use integration by parts and the fact that $\int_{-2}^{2}\sqrt{4-x^2}\,dx = 4\pi$.

One can also use a computer.

10. Recall that the surface $z^2 = x^2 + y^2$ is an hour glass. Since our solid is bounded in the first octant, we only need a quarter of its upper inverted cone. The region is therefore the one depicted in the picture below.



Thus, the integral may be written as

$$\int_0^1 \int_0^{1-x} \int_0^{\sqrt{x^2+y^2}} (2x-1) y z^2 \, dz\, dy\, dx = -\frac{1}{126}$$

according to Mathematica.

11. This is similar to previously solved problems (see problem 8).

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx = \int_0^1 \int_0^{1-x} (1-x-y)\, dy\, dx = \frac{1}{6}$$

12. $$\int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} dz\, dy\, dx = \frac{4}{3}\pi r^3$$

13. The solid's vertical shadow on the xy-plane is the disc $x^2+y^2 \le a^2$. For a given point (x,y) in the disc D, $z^2 \le a^2 - x^2$. In other words,

$$-\sqrt{a^2-x^2} \le z \le \sqrt{a^2-x^2}$$

Hence, the volume of this solid is given by

$$\iint_D \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx =$$

$$= \frac{16a^3}{3}$$

14. $$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} k|z|\,dz\,dy\,dx = \left(\int_{-1}^{1} dx\right)\left(\int_{-1}^{1} dy\right)\left(\int_{-1}^{1} k|z|\,dz\right) =$$

$$= 8\int_0^1 kz\,dz = 4k.$$

15. $$\int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx = \frac{1}{720}$$

16. $$\int_1^2\int_1^2\int_{-2x-3y}^{x+y} k|z|\,dz\,dy\,dx =$$

$$= \int_1^2\int_1^2\int_0^{x+y} kz\,dz\,dy\,dx - \int_1^2\int_1^2\int_{-2x-3y}^{0} kz\,dz\,dy\,dx. =$$

$$= \frac{133k}{4}.$$

17. The solid S is a box. Notice that the iterated integral presents S as a y-simple solid. To make it z-simple, write

$$\iiint_S f(x,y,z) = \iiint_{[-1,0]\times[1,2]\times[0,1]} f(x,y,z) = \iint_{[-1,0]\times[1,2]} \int_0^1 f(x,y,z)\,dz$$

$$= \int_{-1}^{0}\int_{1}^{2}\int_{0}^{1} f(x,y,z)\,dz\,dy\,dx.$$

To make the iterated integral x-simple, write

$$\iiint_{[-1,0]\times[1,2]\times[0,1]} f(x,y,z) = \iint_{[1,2]\times[0,1]} \int_{-1}^{0} f(x,y,z)\,dx =$$

$$= \int_{1}^{2}\int_{0}^{1}\int_{-1}^{0} f(x,y,z)\,dx\,dz\,dy.$$

18. The solid S is a tetrahedron written as a z-simple iterated integral. To represent it as an x-simple solid, observe that $z = 8-4x-2y$ implies that $x = \frac{1}{4}(8-2y-z)$. Hence

$$\iiint_S f(x,y,z) = \iint_{\pi_{(y,z)}(S)} \int_0^{\frac{1}{4}(8-2y-z)} f(x,y,z)\,dx$$

where $\pi_{(y,z)}(x,y,z) = (0,y,z)$ is the projection (shadow) on the yz-plane. The boundaries of $\pi_{(y,z)}(S)$ are the lines $y=0$, $z=0$ and $0 = \frac{1}{4}(8-2y-z)$ or $z = 8-2y$

Thus $\iiint\limits_S f(x,y,z) = \int_0^4 \int_0^{8-2y} \int_0^{\frac{1}{4}(8-2y-z)} f(x,y,z)\,dx\,dz\,dy$

The same approach can be used to produce a y-simple integral. Let us, however modify this approach to illustrate a rather useful trick.

Notice that $z = 8-4x-2y$ can be written as $w = 8-\cancel{2u}-v$ in the uvw-coordinate system, where $u = 4x$, $v = 2y$, and $w = z$. The boundary planes $x=0$, $y=0$, and $z=0$ are the same as $u=0$, $v=0$, $w=0$. Therefore, writing $\iiint\limits_S f$ as a y-simple integral is essentially the same as writing it as a v-simple integral. $v = 8-u-w$ or $2y = 8-4x-z$ so $0 \le y \le \frac{1}{2}(8-4x-z)$. Projecting $v = 8-u-w$ onto the uw-plane we get $0 = 8-u-w$ or $u+w = 8$ from which it follows that

$0 \le u \le 8$ and $0 \le w \le 8-u$ or

$0 \le \cancel{4}x \le 8$ and $0 \le z \le 8-4x$ which simplifies to

$0 \le x \le 2$ and $0 \le z \le 8-4x$. Thus, the desired integral is

$$\int_0^2 \int_0^{8-4x} \int_0^{\frac{1}{2}(8-4x-z)} f(x,y,z)\,dy\,dz\,dx.$$

19. Notice that the integral is set as y-simple, with

$$0 \le y \le 3\sqrt{1-x^2-z^2/4}$$
$$0 \le x \le \sqrt{1-z^2/4}$$
$$0 \le z \le 2$$

The upper boundary of y is $y = 3\sqrt{1-x^2-z^2/4}$ or $\frac{y^2}{9} = 1-x^2-z^2/4$, which simplifies to $x^2+\frac{y^2}{9}+\frac{z^2}{4}=1$. Since $x,y,z \geq 0$, we see that the region of integration is the quarter ellipsoid in the first octant.

To write the integral as z-simple, we solve $x^2+\frac{y^2}{9}+\frac{z^2}{4}=1$ for z obtaining

$$0 \le z \le 2\sqrt{1-x^2-\tfrac{y^2}{9}}.$$

Therefore

$$\iiint_S f(x,y,z) = \iint_{\Pi_{(x,y)}(S)} \int_0^{2\sqrt{1-x^2-y^2/9}} f(x,y,z)\,dz$$

where $\Pi_{(x,y)}$ is the projection on the xy-plane (i.e. $\Pi_{(x,y)}(x,y,z) = (x,y,0)$. Notice that $\Pi_{(x,y)}$ is a linear map). Observe that $\Pi_{(x,y)}(S) = \mathcal{E}$ where $\mathcal{E}$ is the elliptical disc $x^2+\frac{y^2}{9}=1$

Thus, we may set this integral as

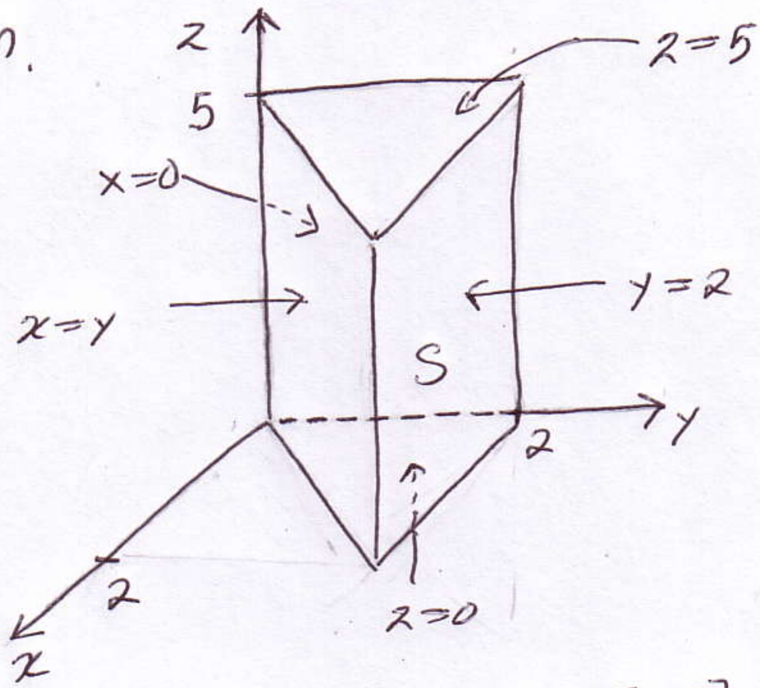
$$\int_0^1 \int_0^{3\sqrt{1-x^2}} \int_0^{2\sqrt{1-x^2-y^2/4}} f(x,y,z)\, dz\, dy\, dx.$$

You should verify that the setting

$$\int_0^3 \int_0^{2\sqrt{1-y^2/9}} \int_0^{\sqrt{1-y^2/9-z^2/4}} f(x,y,z)\, dx\, dz\, dy$$

makes the integral x-simple.

20. The solid S over which we are integrating looks like a triangular slice of cheese. To be more specific, it is a tetrahedron.



Notice that $\Pi_{(y,z)}(S) = [0,2] \times [0,5]$ and that

$$0 \leq x \leq y$$

Hence $\int_0^5 \int_0^2 \int_0^y f(x,y,z)\, dx\, dy\, dz$ is an x-simple representation of the integral.

To write the integral as y-simple, notice that

$\Pi_{(x,z)}(S) = [0,2] \times [0,5]$ and that

$$x \le y \le 2$$

Hence,

$$\int_0^5 \int_0^2 \int_x^2 f(x,y,z)\, dy\, dx\, dz \quad \text{is a } y\text{-simple}$$

representation.