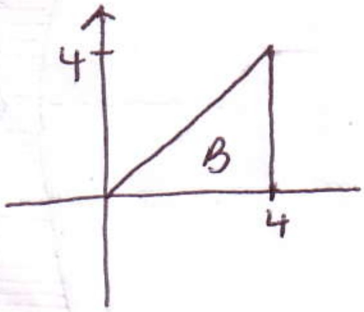
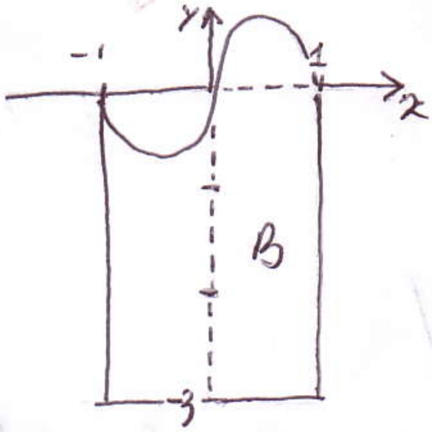
Solutions to H.W #14

1.



B is both x- and y-simple.

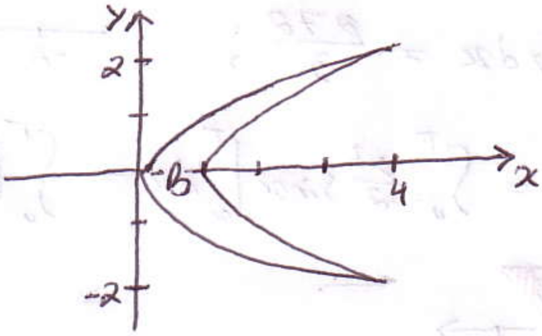
2.



B is y-simple.

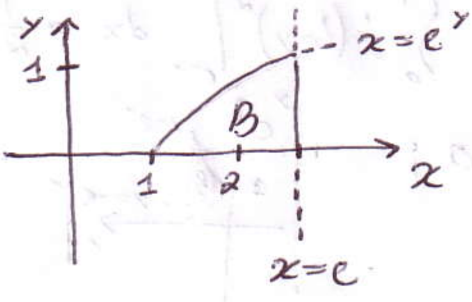
3. $y^2 = \frac{3}{4}y^2 + 1$ whenever $\frac{1}{4}y^2 = 1$ or $y = \pm 2$.

Thus $B = \{(x,y) : y \in [-2,2],\ y^2 \leq x \leq \frac{3}{4}y^2 + 1\}$



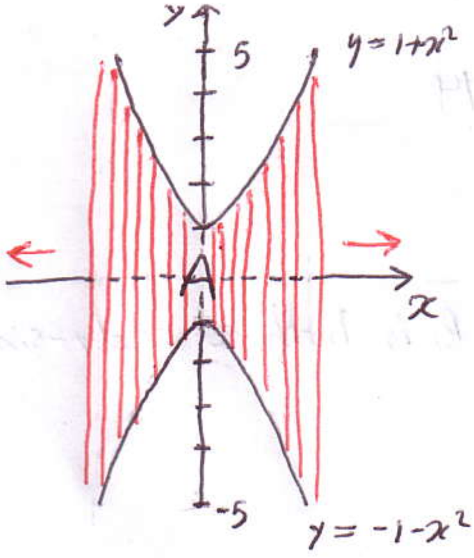
B is x-simple

4.

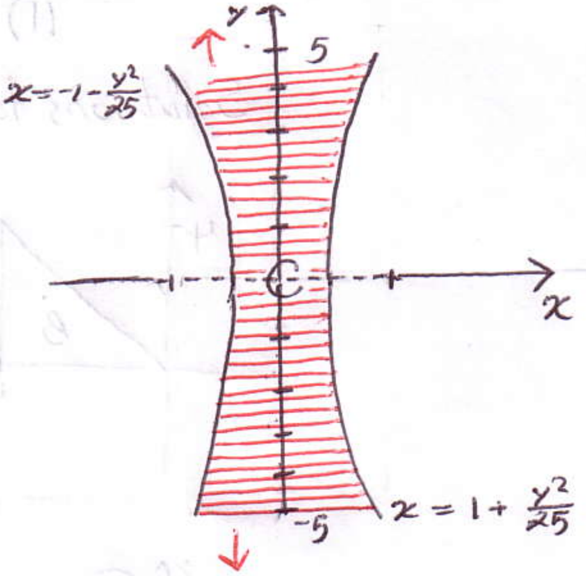


B is both x- and y-simple

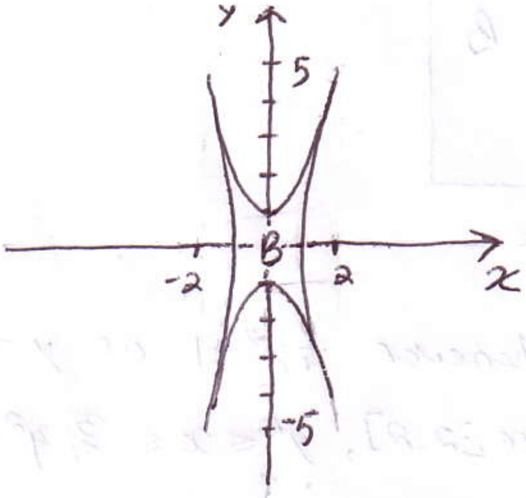
5.



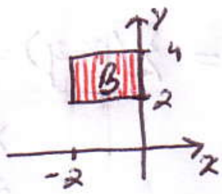
(2)



Thus the desired region is



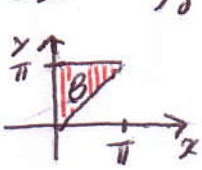
B is neither x- nor y-simple



6. $\int_{-2}^{0}\int_{2}^{4}(3x^2+2y^2)\,dy\,dx = \frac{272}{3}$;

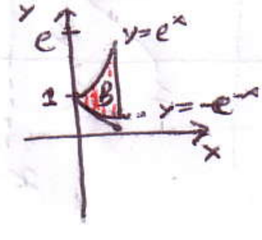
7. $\int_0^{\pi}\int_x^{\pi} y\sin x\,dy\,dx = \int_0^{\pi}\frac{y^2}{2}\sin x\Big|_x^{\pi}dx = \int_0^{\pi}\left(\frac{\pi^2}{2}\sin x - \frac{x^2}{2}\sin x\right)dx =$

$= \frac{1}{2}(4+\pi^2)$;



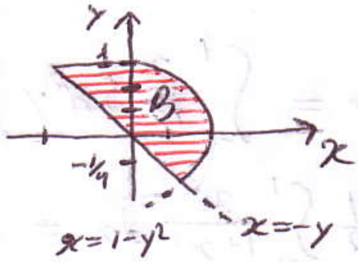
8. $\int_0^1\int_{e^{-x}}^{e^x}\frac{\ln y}{y}\,dy\,dx = \int_0^1\frac{1}{2}(\ln y)^2\Big|_{e^{-x}}^{e^x}dx = \frac{1}{2}\int_0^1\left((\ln e^x)^2-(\ln e^{-x})^2\right)dx =$

$= \frac{1}{2}\int_0^1(x^2-x^2)\,dx = 0$;



9. $\int_{-1/4}^{1}\int_{-y}^{1-y^2}(5x-y)\,dx\,dy = \int_{-1/4}^{1}\left(\frac{5}{2}x^2 - xy\right)\Big|_{-y}^{1-y^2} dy =$

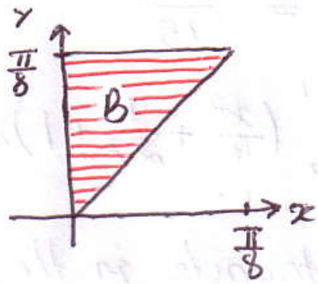
$= \int_{-1/4}^{1}\left(\frac{5}{2}(1-y^2)^2 - y(1-y^2) - \frac{5}{2}y^2 - y^2\right)dy = \frac{3245}{6144}$;



10. $\int_0^{\frac{\pi}{8}}\int_0^{y}\sec^2(x+y)\,dx\,dy = \int_0^{\pi/8}\tan(x+y)\Big|_0^{y} dy =$

$= -\frac{1}{2}\ln(\cos 2y) + \ln(\cos y)\Big|_0^{\frac{\pi}{8}} = \ln\left(\cos\frac{\pi}{8}\right) - \frac{1}{2}\ln\left(\cos\frac{\pi}{4}\right) =$

$= \ln\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right) - \frac{1}{2}\ln\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}\ln\left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right)$;



11. $\int_{-1}^{5}\int_{-1}^{2}(x-2y)^2\,dy\,dx = 126$

12. $x^2 = 4x$ whenever $x=0$ or $x=4$. Thus

$\int_0^4\int_{x^2}^{4x}\frac{x}{(y+1)^2}\,dy\,dx = \int_0^4 \frac{-x}{y+1}\Big|_{x^2}^{4x} dx = \int_0^4\left(\frac{-x}{4x+1} + \frac{x}{x^2+1}\right)dx$

$= \frac{9\ln(17)}{16} - 1$

13. $y^2 = 5y$ whenever $y=0$ or $y=5$. Thus

$$\int_0^5 \int_{y^2}^{5y} y e^{x}\, dx\, dy = \int_0^5 y\left(e^{5y} - e^{y^2}\right) dy = \frac{1}{50}\left(27 + 23e^{25}\right)$$

14. $$\int_{-1}^{1} \int_0^{\tan^{-1}(x)} x \cos y \sin y \, dy\, dx = \int_{-1}^{1} \frac{x}{2} \sin^2 y \Big|_0^{\tan^{-1} x} dx =$$

$$= \int_{-1}^{1} \frac{x}{2} \sin^2(\tan^{-1} x)\, dx = \frac{1}{2}\int_{-1}^{1} \frac{x^2}{1+x^2}\, dx = 1 - \frac{\pi}{4}$$

15. $$\int_0^{\ln 2}\int_0^1 e^x\, dy\, dx = \int_0^{\ln 2} y e^x \Big|_0^1 dx = \int_0^{\ln 2} e^x\, dx = e^{\ln 2} - e^0 = 1$$

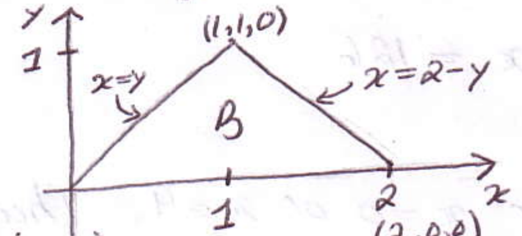
16. $$\int_0^{\pi}\int_0^{\sin x} (\cos x - y)\, dy\, dx = \int_0^{\pi} \left(y\cos x - \frac{y^2}{2}\right)\Big|_0^{\sin x} dx =$$

$$= \int_0^{\pi} \left(\sin x \cos x - \frac{\sin^2 x}{2}\right) dx = -\frac{\pi}{4}$$

17. $$\int_0^2 \int_0^{1+y^2} x\, dx\, dy = \int_0^2 \frac{(1+y^2)^2}{2}\, dy = \frac{103}{15}$$

18. $$\int_0^1\int_0^1 \left(\frac{x^2}{4} + \frac{y^2}{9} + 1\right) dy\, dx = \int_0^1 \left(\frac{x^2}{4} + \frac{1}{27} + 1\right) dx = \frac{121}{108}$$

19. The region of integration is the triangle in the picture below.



This region is both x- and y-simple. Treating it as x-simple, the region B is given by $\{(x,y): y \in [0,1],\ y \le x \le 2-y\}$. Since our solid is bounded above by $z=0$ and below by $z = -8 + x + \frac{y}{2}$.

Thus the volume is given by

$$\int_0^1\int_y^{2-y}\left(8-x-\frac{y}{2}\right)dx\,dy=\frac{41}{6}$$

20. $(1-x^2)(1-y^2)=0$ whenever $x=\pm1$, $y=\pm1$. Thus, the region of integration is bounded by $x=1$, $x=-1$, $y=1$, $y=-1$.

In particular, $B=[-1,1]\times[-1,1]$.

The desired volume is given by

$$\int_{-1}^1\int_{-1}^1(1-x^2)(1-y^2)\,dx\,dy=\left(\int_{-1}^1(1-x^2)\,dx\right)\left(\int_{-1}^1(1-y^2)\,dy\right)=$$

$$=\left(\int_{-1}^1(1-x^2)\,dx\right)^2=\left(2\int_0^1(1-x^2)\,dx\right)^2=4\left(1-\frac{1}{3}\right)^2=\frac{16}{9}.$$

21. Since the region is bounded between $z=1$ and $z=x^2+y^2$,

The region of integration is given by $B=\{(x,y): x^2+y^2=1\}=$

$=\{(x,y): x\in[-1,1],\ -\sqrt{1-x^2}\le y\le\sqrt{1-x^2}\}.$

Thus, the desired integral is

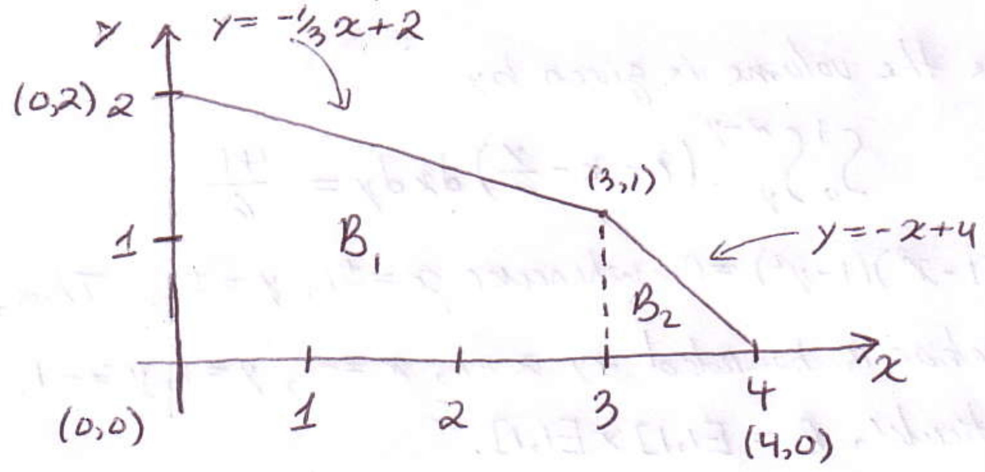
$$\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\,dx=\int_{-1}^1 2\int_0^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\,dx=$$

$$=4\int_0^1\int_0^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\,dx=4\int_0^1\left(\sqrt{1-x^2}-x^2\sqrt{1-x^2}-\frac{\left(\sqrt{1-x^2}\right)^3}{3}\right)dx=$$

$$=4\frac{\pi}{8}=\frac{\pi}{2}$$

Remark: Discs and Washers method from Calc I would have been more effective in this case.

22.



$B = B_1 \cup B_2$. Therefore $A(B) = A(B_1 \cup B_2) = A(B_1) + A(B_2) =$

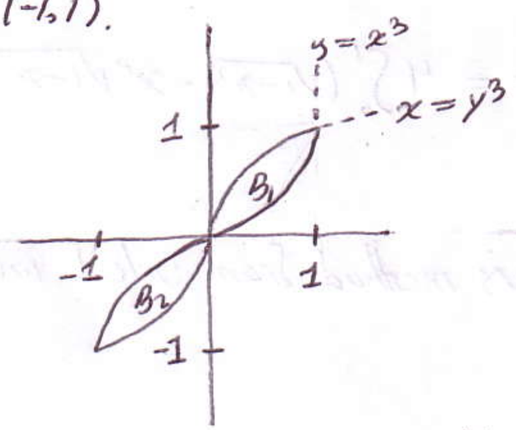
$$= \int_0^3 \int_0^{-\frac{1}{3}x+2} 1\, dy\, dx + \int_3^4 \int_0^{-x+4} 1\, dy\, dx = \int_0^3 (-\tfrac{1}{3}x+2)\, dx +$$

$$+ \int_3^4 (-x+4)\, dx = \frac{-9}{6} + 6 + \frac{1}{2} = 5$$

23. $D = \{(x,y) : x^2+y^2=100\} = \{(x,y) : x \in [-10,10], -\sqrt{100-x^2} \leq y \leq \sqrt{100-x^2}$

Therefore $A(D) = \int_{-10}^{10} \int_{-\sqrt{100-x^2}}^{\sqrt{100-x^2}} 1\, dy\, dx = 100\pi$.

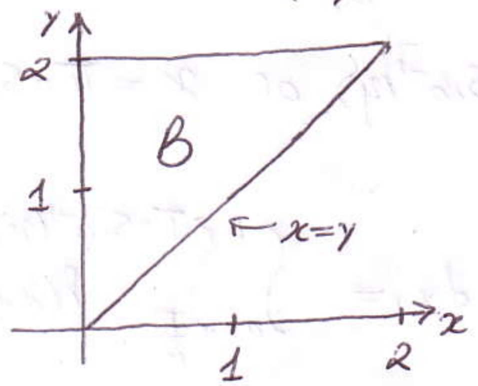
24. The curves $y = x^3$ and $x = y^3$ intersect at the points $(0,0)$, $(1,1)$, and $(-1,1)$.



$B = B_1 \cup B_2$ where $A(B_1) = A(B_2)$. Hence $A(B) = A(B_1) + A(B_2) =$

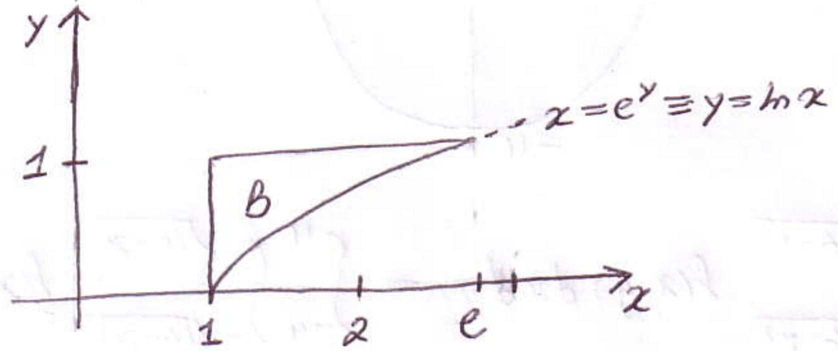
$$= 2A(B_1) = 2 \int_0^1 \int_{x^3}^{\sqrt[3]{x}} 1\, dy\, dx = 2 \int_0^1 (\sqrt[3]{x} - x^3)\, dx = 1.$$

25.



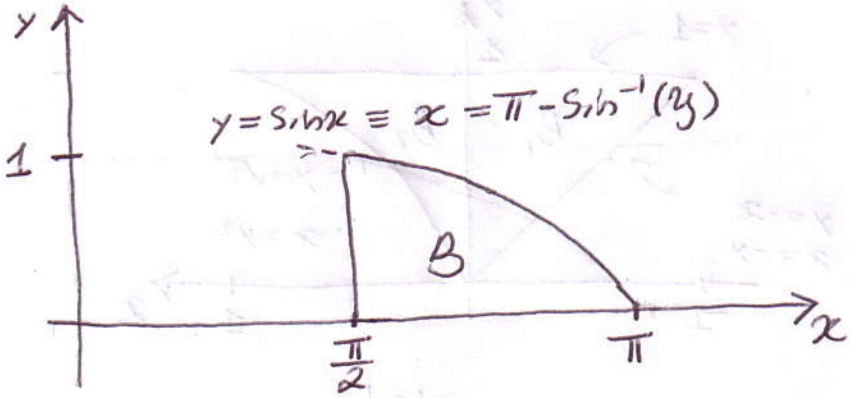
$$\int_0^2\int_x^2 f(x,y)\,dy\,dx = \int_0^2\int_0^y f(x,y)\,dx\,dy$$

26.



$$\int_0^1\int_1^{e^y} f(x,y)\,dx\,dy = \int_1^e\int_{\ln x}^1 f(x,y)\,dy\,dx$$

27.

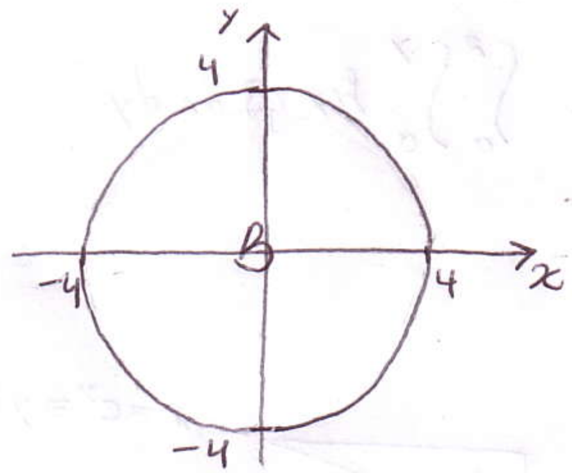


To see that $y = \sin x$, $x \in [\frac{\pi}{2}, \pi] \Rightarrow x = \pi - \sin^{-1}(y)$, let $u = x - \pi$. Then $y = \sin x = \sin(x - \pi + \pi) = \sin(u + \pi) =$ $= -\sin u$ where $u \in [-\frac{\pi}{2}, 0]$. Hence $u = \sin^{-1}(-y) = -\sin^{-1} y$

In particular, $x - \pi = -\sin^{-1}(y)$ or $x = \pi - \sin^{-1}(y)$ as desired.

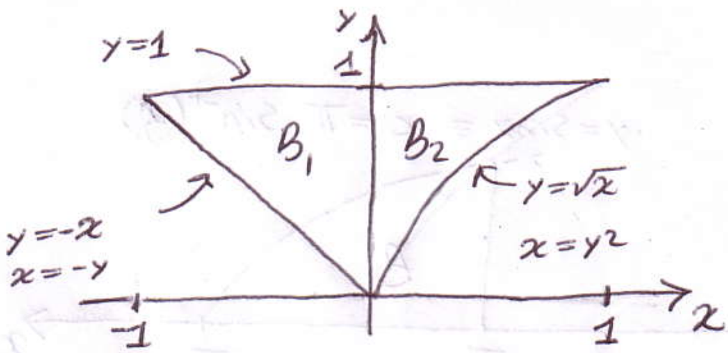
Thus $$\int_{\frac{\pi}{2}}^{\pi}\int_{0}^{\sin x} f(x,y)\,dy\,dx = \int_{0}^{1}\int_{\frac{\pi}{2}}^{\pi-\sin^{-1}(y)} f(x,y)\,dx\,dy$$

28.



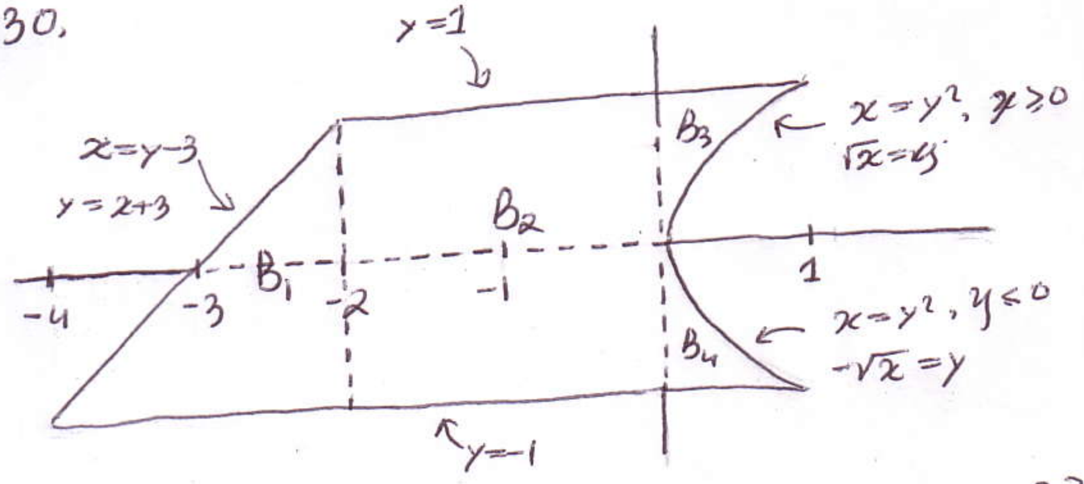
$$\int_{-4}^{4}\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} f(x,y)\,dx\,dy = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} f(x,y)\,dy\,dx.$$

29. Observe that $B = B_1 \cup B_2$, where $B_1 = \{(x,y) : x \in [-1,0], -x \le y \le 1\}$ and $B_2 = \{(x,y) : x \in [0,1], \sqrt{x} \le y \le 1\}$



Thus $$\int_{-1}^{0}\int_{-x}^{1} f(x,y)\,dy\,dx + \int_{0}^{1}\int_{\sqrt{x}}^{1} f(x,y)\,dy\,dx = \int_{0}^{1}\int_{-y}^{y^2} f(x,y)\,dx\,dy.$$

30.



Observe that $B = \{(x,y) : y \in [-1,1],\ y-3 \le x \le y^2\}$ is x-simple but not y-simple. We can break B into a union of y-simple regions $B_1 = \{(x,y) : x \in [-4,-2],\ -1 \le y \le x+3\}$,

$B_2 = \{(x,y) : x \in [-2,0],\ -1 \le y \le 1\}$, $B_3 = \{(x,y) : x \in [0,1],\ \sqrt{x} \le y \le 1\}$,

and $B_4 = \{(x,y) : x \in [0,1],\ -1 \le y \le -\sqrt{x}\}$.

Thus $\int_{-1}^{1}\int_{y-3}^{y^2} f(x,y)\,dx\,dy = \iint_B f = \iint_{B_1 \cup B_2 \cup B_3 \cup B_4} f =$

$$= \int_{-4}^{-2}\int_{-1}^{x+3} f(x,y)\,dy\,dx + \int_{-2}^{0}\int_{-1}^{1} f(x,y)\,dy\,dx + \int_{0}^{1}\int_{\sqrt{x}}^{1} f(x,y)\,dy\,dx +$$

$$+ \int_{0}^{1}\int_{-1}^{-\sqrt{x}} f(x,y)\,dy\,dx.$$