Solutions to H.W #12

1. $\nabla f(x,y) = (4x-3y,\ -3x+28y)$

$$H_{f(x,y)} = \begin{pmatrix} 4 & -3 \\ -3 & 28 \end{pmatrix} \quad \text{Thus } H_{f(1,4)} = \begin{pmatrix} 4 & -3 \\ -3 & 28 \end{pmatrix}$$

2. $\nabla f(x,y) = \left(\frac{1}{x+y},\ \frac{1}{x+y}\right)$

$$H_{f(x,y)} = \begin{pmatrix} -1/(x+y)^2 & -1/(x+y)^2 \\ -1/(x+y)^2 & -1/(x+y)^2 \end{pmatrix} \quad \text{Thus } H_{f(1,1)} = \begin{pmatrix} -1/4 & -1/4 \\ -1/4 & -1/4 \end{pmatrix}$$

3. $\nabla f(x,y,z) = \left(1/{1+x^2},\ z,\ y\right)$

$$H_{f(x,y,z)} = \begin{pmatrix} \frac{-2x}{(1+x^2)^2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{Thus } H_{f(0,3,1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

4. $\nabla f(x_1,x_2,x_3,x_4) = \left(\frac{1}{x_3+x_4},\ \frac{1}{x_3+x_4},\ \frac{-(x_1+x_2)}{(x_3+x_4)^2},\ \frac{-(x_1+x_2)}{(x_3+x_4)^2}\right)$

$$H_{f(x_1,x_2,x_3,x_4)} = \begin{pmatrix} 0 & 0 & \frac{-1}{(x_3+x_4)^2} & \frac{-1}{(x_3+x_4)^2} \\ 0 & 0 & \frac{-1}{(x_3+x_4)^2} & \frac{-1}{(x_3+x_4)^2} \\ \frac{-1}{(x_3+x_4)^2} & \frac{-1}{(x_3+x_4)^2} & \frac{2(x_1+x_2)}{(x_3+x_4)^3} & \frac{2(x_1+x_2)}{(x_3+x_4)^3} \\ \frac{-1}{(x_3+x_4)^2} & \frac{-1}{(x_3+x_4)^2} & \frac{2(x_1+x_2)}{(x_3+x_4)^3} & \frac{2(x_1+x_2)}{(x_3+x_4)^3} \end{pmatrix}$$

Thus $H_{f(1,1,1,1)} = \begin{pmatrix} 0 & 0 & -1/4 & -1/4 \\ 0 & 0 & -1/4 & -1/4 \\ -1/4 & -1/4 & 1/2 & 1/2 \\ -1/4 & -1/4 & 1/2 & 1/2 \end{pmatrix}$

5. $\nabla f(x_1, x_2, \dots, x_n) = (1, 1, \dots, 1)$

$H_{f(x_1, x_2, \dots, x_n)} = O_n$, where O_n is the $n\times n$ matrix with all entries $o_{ij} = 0$ (zero).

6. $f(x) = \|x\|^2$; $x \in \mathbb{R}^n$. $\nabla f(x) = (2x_1, \dots, 2x_n)$

$H_{f(x)} = 2I_n$ where I_n is the $n\times n$ identity matrix.

Thus, for $a \in \mathbb{R}^n$ $H_{f(a)} = 2I_n$.

7. $f(1,-1) = 7-4-3 = 0$

$(x-1, y+1)\cdot \nabla f\big|_{(1,-1)} = (x-1)\frac{\partial f}{\partial x}(1,-1) + (y+1)\frac{\partial f}{\partial y}(1,-1) =$

$= 10(x-1) + 10(y+1)$

$\frac{1}{2}\left((x-1, y+1)\cdot\nabla\right)^2 f\big|_{(1,-1)} = \left[(x-1)^2 \frac{\partial^2 f}{\partial x^2}(1,-1) + 2(x-1)(y+1)\frac{\partial^2 f}{\partial x \partial y}(1,-1)\right.$

$\left. + (y+1)^2 \frac{\partial^2 f}{\partial y^2}(1,-1)\right] = \frac{1}{2}\left(14(x-1)^2 + 8(x-1)(y+1) - 6(y+1)^2\right) =$

$= 7(x-1)^2 + 4(x-1)(y+1) - 3(y+1)^2$

Thus $P_2(x,y) = 10(x-1) + 10(y+1) + 7(x-1)^2 + 4(x-1)(y+1) - 3(y+1)^2$

8. $f(2,3) = 8 + 27 = 35$

$$(x-2, y-3)\cdot \nabla f\Big|_{(2,3)} = (x-2)\frac{\partial f}{\partial x}(2,3) + (y-3)\frac{\partial f}{\partial y}(2,3) =$$

$$= 12(x-2) + 27(y-3)$$

$$\frac{1}{2}\Big((x-2, y-3)\cdot\nabla\Big)^2 f\Big|_{(2,3)} =$$

$$= \frac{1}{2}\left((x-2)^2\frac{\partial^2 f}{\partial x^2}(2,3) + 2(x-2)(y-3)\frac{\partial^2 f}{\partial x\partial y}(2,3) + (y-3)^2\frac{\partial^2 f}{\partial y^2}(2,3)\right)$$

$$= \frac{1}{2}\left((x-2)^2\cdot 12 + 2(x-2)(y-3)\cdot 0 + (y-3)^2\cdot 18\right) =$$

$$= 6(x-2)^2 + 9(y-3)^2$$

Thus $P_2(x,y) = 35 + 12(x-2) + 27(y-3) + 6(x-2)^2 + 9(y-3)^2$.

9. $\nabla f(x,y) = \left(2x\cos(x^2+y^2),\ 2y\cos(x^2+y^2)\right)$; $\nabla f(0,0) = (0,0)$

Also $f(0,0) = 0$.

Observe that $\frac{1}{2}((x,y)\cdot\nabla)^2 f\Big|_{(0,0)} \equiv \frac{1}{2}(x\ y)\,H_{f(0,0)}\begin{pmatrix}x\\ y\end{pmatrix}$

Now $H_{f(x,y)} = \begin{pmatrix} 2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2) & -4xy\sin(x^2+y^2) \\ -4xy\sin(x^2+y^2) & 2\cos(x^2+y^2) - 4y^2\sin(x^2+y^2)\end{pmatrix}$

so $H_{f(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 2\end{pmatrix}$ and $\frac{1}{2}(x\ y)\,H_{f(0,0)}\begin{pmatrix}x\\ y\end{pmatrix} =$

$$= (x\ y)\begin{pmatrix}x\\ y\end{pmatrix} = (x^2+y^2) \equiv x^2+y^2$$

Hence $P_2(x,y) = x^2 + y^2$

10. $f(1,1,0) = 1+1 = 2$

$\nabla f(x,y,z) = (1, e^z, ye^z)$; $\nabla f(1,1,0) = (1,1,1)$

$(x-1, y-1, z) \cdot \nabla f(1,1,0) = x-1+y-1+z = x+y+z-2$.

Now $\frac{1}{2}\left((x-1, y-1, z)\cdot \nabla\right)^2 f\Big|_{(1,1,0)} \equiv \frac{1}{2}(x-1 \;\; y-1 \;\; z)\, H_{f(1,1,0)} \begin{pmatrix} x-1 \\ y-1 \\ z \end{pmatrix}$

$$H_{f(x,y,z)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^z \\ 0 & e^z & ye^z \end{pmatrix} \text{ so } H_{f(1,1,0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and $\frac{1}{2}(x-1 \;\; y-1 \;\; z) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x-1 \\ y-1 \\ z \end{pmatrix} =$

$= \frac{1}{2}(x-1 \;\; y-1 \;\; z) \begin{pmatrix} 0 \\ z \\ y-1 \end{pmatrix} = \frac{1}{2}\left((x-1)z + 2(y-1)z\right) \equiv$

$\frac{1}{2}xz - \frac{1}{2}z + yz - z = \frac{1}{2}xz + yz - \frac{3}{2}z$

Thus $P_2(x,y,z) = x + y - \frac{1}{2}z + \frac{1}{2}xz + yz$.

11. $f(1,1,1) = \frac{\pi}{4}$ $\quad \nabla f(x,y,z) = \left(\frac{-yz}{x^2}\frac{1}{1+(\frac{y}{x})^2}, \frac{z}{x}\frac{1}{1+(\frac{y}{x})^2}, \tan^{-1}\left(\frac{y}{x}\right)\right)$

$= \left(\frac{-yz}{x^2+y^2}, \frac{xz}{x^2+y^2}, \tan^{-1}\left(\frac{y}{x}\right)\right)$; $\nabla f(1,1,1) = \left(-\frac{1}{2}, \frac{1}{2}, \frac{\pi}{4}\right)$

so $(x-1, y-1, z-1)\cdot \nabla f(1,1,1) = -\frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{\pi}{4}(z-1)$;

$$H_{f(x,y,z)} = \begin{pmatrix} \frac{2xyz}{(x^2+y^2)^2} & -\frac{(x^2+y^2)z + 2y^2z}{(x^2+y^2)^2} & \frac{-y}{x^2+y^2} \\ \frac{(x^2+y^2)z - 2x^2z}{(x^2+y^2)^2} & \frac{-2xyz}{(x^2+y^2)^2} & \frac{x}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix}$$

so $H_{f(1,1,1)} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

Consequently, $\frac{1}{2}\left((x-1, y-1, z-1)\cdot\nabla\right)^2 f\Big|_{(1,1,1)} \equiv$

$\frac{1}{2}\begin{pmatrix} x-1 & y-1 & z-1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} =$

$= \frac{1}{4}\begin{pmatrix} x-1 & y-1 & z-1 \end{pmatrix}\begin{pmatrix} (x-1)-(z-1) \\ -(y-1)+(z-1) \\ -(x-1)+(y-1) \end{pmatrix} =$

$= \frac{1}{4}\left((x-1)^2 - 2(x-1)(z-1) - (y-1)^2 + 2(y-1)(z-1)\right)$

Thus, $P_2(x,y,z) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{\pi}{4}(z-1) +$
$+ \frac{1}{4}(x-1)^2 - \frac{1}{2}(x-1)(z-1) - \frac{1}{4}(y-1)^2 + \frac{1}{2}(y-1)(z-1)$

12. Suppose that $p_n(x)$ is the n^{th} degree Taylor polynomial of $g(x)$ centered at 0. Then

$$p_n(x) = \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} x^k$$

If $P_n(x,y)$ is the n^{th} order Taylor polynomial of $f(x,y) = g(ax+by)$ centered at $(0,0)$, then

$$P_n(x,y) = \sum_{k=0}^{n} \frac{\left[(x,y)\cdot\nabla\right]^k f(0,0)}{k!}$$

Thus, it suffices to show that $[(x,y)\cdot\nabla]^k f(0,0) =$
$= g^{(k)}(0)(ax+by)^k$ for all $k\in\{0,\dots,n\}$.

By Clairaut's theorem and the binomial theorem

$$[(x,y)\cdot\nabla]^k f(0,0) = \sum_{i=0}^{k}\binom{k}{i}x^i y^{k-i}\frac{\partial^k f}{(\partial x)^i(\partial y)^{k-i}}(0,0) =$$

$$= \sum_{i=0}^{k}\binom{k}{i}x^i y^{k-i}\frac{\partial^k}{(\partial x)^i(\partial y)^{k-i}}g(ax+by)\Big|_{(x,y)=(0,0)} =$$

$$= \sum_{i=0}^{k}\binom{k}{i}x^i y^{k-i}a^i b^{k-i}g^{(k)}(ax+by)\Big|_{(x,y)=(0,0)} =$$

$$= \sum_{i=0}^{k}\binom{k}{i}(ax)^i(by)^{k-i}g^{(k)}(0) = g^{(k)}(0)\sum_{i=0}^{k}\binom{k}{i}(ax)^i(by)^{k-i}$$

$= g^{(k)}(0)(ax+by)^k$ and the result follows as desired.

13. $f(x,y) = e^{x+y} = g(x+y)$ where $g(x) = e^x$.

Thus, by problem 12, $P_4(x,y) = P_4(x+y) = 1+(x+y)+\frac{(x+y)^2}{2}+$
$+\frac{(x+y)^3}{3!}+\frac{(x+y)^4}{4!}$ and $P_\infty(x,y) = P_\infty(x+y) = \sum_{k=0}^{\infty}\frac{(x+y)^k}{k!}$

14. Let $g(x) = e^{x^2}$. Then $P_n(x) = \sum_{k=0}^{n}\frac{x^{2k}}{k!}$

Since $e^{(x+y)^2} = g(x+y)$, it follows that $P_4(x,y) =$
$= P_4(x+y) = 1+(x+y)^2+\frac{(x+y)^4}{2}+\frac{(x+y)^6}{3!}+\frac{(x+y)^8}{4!}$

Similarly $P_\infty(x,y) = P_\infty(x+y) = \sum_{k=0}^{\infty}\frac{(x+y)^{2k}}{k!}$

15. Let $g(x) = xe^{\frac{x}{4}}$, then $P_n(x) = x\sum_{k=0}^{n} \frac{\left(\frac{x}{4}\right)^k}{k!} =$

$= x\sum_{k=0}^{n} \frac{x^k}{4^k k!} = \sum_{k=0}^{n} \frac{x^{k+1}}{4^k k!}$

Since $f(x,y) = g(x-y)$, it follows that $P_4(x,y) = P_4(x-y) =$

$= \sum_{k=0}^{4} \frac{(x-y)^{k+1}}{4^k k!}$

Similarly, $P_\infty(x,y) = \sum_{k=0}^{\infty} \frac{(x-y)^{k+1}}{4^k k!}$.

16. Notice that $f(x,y) = \frac{1}{1-(x+y)^2}$.

Let $g(x) = \frac{1}{1-x^2} = \sum_{k=0}^{\infty} x^{2k}$. Thus $P_n(x) = \sum_{k=0}^{n} x^{2k}$.

Since $f(x,y) = g(x+y)$, $P_6(x,y) = P_6(x+y) = \sum_{k=0}^{6} (x+y)^{2k}$

and $P_\infty(x,y) = P_\infty(x+y) = \sum_{k=0}^{\infty} (x+y)^{2k}$