(6.2)

Double and triple integrals are frequently difficult to set up and to evaluate. In this section, we would like to develop a procedure analogous to the u-substitution method of single-variable calculus that can transform many cumbersome integrals into more managable expressions.

Before we begin, it would be helpful to review the area and volume properties associated with linear transformations.

Let $S: \mathbb{R}^2 \to \mathbb{R}^2$ be a one-to-one and onto linear map and let $R = [x_1, x_2] \times [y_1, y_2]$ be an arbitrary rectangle.

What is the relationship between $V(R) = (x_2 - x_1)(y_2 - y_1) =$
$= \Delta x \Delta y$ and $V(S(R))$, the area of $S(R)$?

Since S is linear, S is of the form $S(x, y) = (ax + by, cx + dy) \equiv$

$$\equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{M}(S) x^T.$$

Notice that $R = [x_1, x_1 + \Delta x] \times [y_1, y_1 + \Delta y] =$
$= \{(x, y) \in \mathbb{R}^2 : (x, y) = (x_1, y_1) + s\Delta x \vec{e}_1 + t\Delta y \vec{e}_2,\ s, t \in [0, 1]\}$

Consequently,

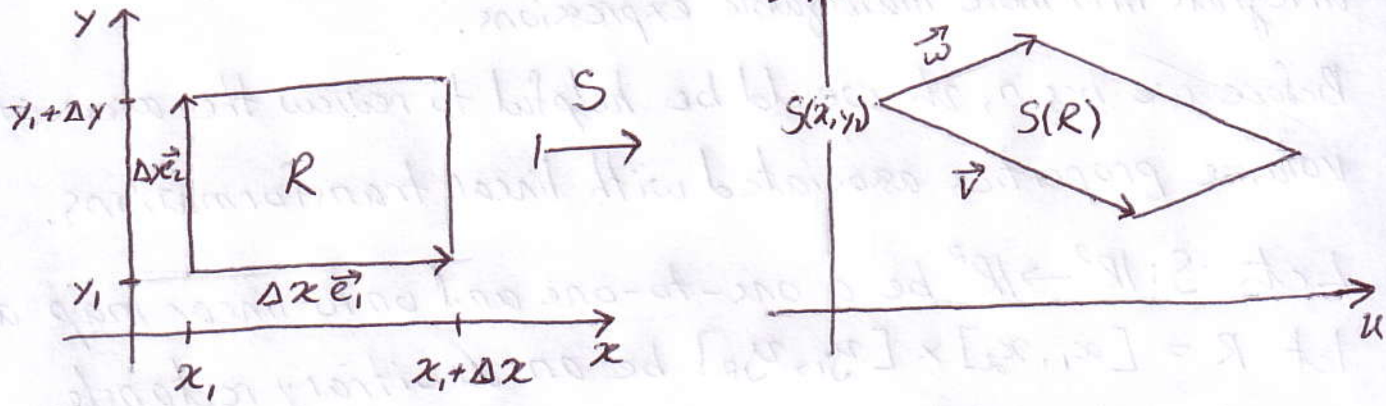
$S(R) = \{(u, v) \in \mathbb{R}^2 : (u, v) = S(x_1, y_1) + s\Delta x S(\vec{e}_1) + t\Delta y S(\vec{e}_2)\}$

where we used the fact that

$$S\left((x_1, y_1) + s\Delta x \vec{e}_1 + t\Delta y \vec{e}_2\right) = S(x_1, y_1) + s\Delta x\, S(\vec{e}_1) + t\Delta y\, S(\vec{e}_2)$$

(why is that so?)

In particular, it follows that $S(R)$ is a parallelogram with corner $S(x_1, y_1)$ and spanned by the vectors $\vec{v} = \Delta x\, S(\vec{e}_1)$ and $\vec{w} = \Delta y\, S(\vec{e}_2)$.



Therefore,

$$V(S(R)) = \left|\det\begin{pmatrix}\vec{v}\\ \vec{w}\end{pmatrix}\right| = \left|\det\begin{pmatrix}\Delta x\, S(\vec{e}_1)\\ \Delta y\, S(\vec{e}_2)\end{pmatrix}\right| = \left|\Delta x \Delta y \det\begin{pmatrix}S(\vec{e}_1)\\ S(\vec{e}_2)\end{pmatrix}\right| =$$

$$= \Delta x \Delta y \left|\det\begin{pmatrix}a & c\\ b & d\end{pmatrix}\right| = \Delta x \Delta y \left|\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}^T\right| =$$

$$= \left|\det \mathcal{M}(S)^T\right| \Delta x \Delta y = \left|\det \mathcal{M}(S)\right| V(R),$$

where we used the fact that the determinant of a square matrix is equal to the determinant of its transpose (see optional handout on determinants).

Remark: Note that this implies that for any region $B \subset \mathbb{R}^2$ with area $V(B)$, The area of $S(B)$, $V(S(B)) = |\det \mathcal{M}(S)|\, V(B)$. (why?)

Similarly, if $S: \mathbb{R}^3 \to \mathbb{R}^3$ is one-to-one and onto linear map and $K = [x_1, x_1+\Delta x] \times [y_1, y_1+\Delta y] \times [z_1, z_1+\Delta z]$, then $S(K)$ is a parallelopiped whose volume $V(S(K)) = |\det \mathcal{M}(S)| V(K) =$
$= |\det \mathcal{M}(S)| \Delta x \Delta y \Delta z$ as you should verify.

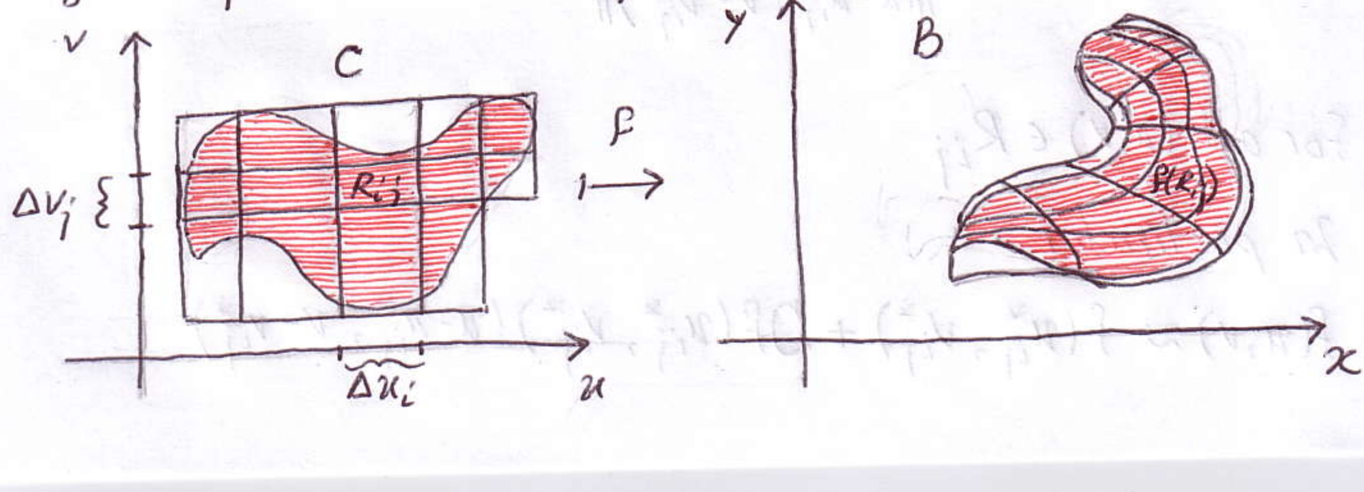
Remark: This implies that if $\mathcal{U} \subset \mathbb{R}^3$ is a solid with volume $V(\mathcal{U})$, $S(\mathcal{U})$ is a solid with volume $V(S(\mathcal{U})) = |\det \mathcal{M}(S)| V(\mathcal{U})$ (why?)

Finally, recall that a linear map $S: \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one and onto iff S is invertible iff $\mathcal{M}(S)$ is invertible iff $\det \mathcal{M}(S) \neq 0$.

We are now ready for the main results.

Change of Variables in Double Integrals

Consider a double integral $\iint_B g$, where g is a continuous function and B is a region in the xy-plane that is the image of a region C in the uv-plane under a smooth one-to-one function $f: C \to B$. In other words, let B be parametrized by $(x,y) = f(u,v)$ for $(u,v) \in C$. Let P be a partition of C into rectangles R_{ij} with area $\Delta u_i \Delta v_j$.



If the partition P is fine (i.e. $N(P)$ is small), then

$$B = f(C) \approx f\Big(\bigcup_{R_{ij} \cap C \neq \emptyset} R_{ij}\Big) = \bigcup_{R_{ij} \cap C \neq \emptyset} f(R_{ij}).$$

In other words, if the "pixel rendering" of C is sufficiently similar to C, then the image of the pixel rendering under f will resemble the image of C under f.

Thus,

$$\iint_B g = \iint_{f(C)} g \approx \iint_{\substack{\bigcup f(R_{ij}) \\ R_{ij} \cap C \neq \emptyset}} g = \sum_{R_{ij} \cap C \neq \emptyset} \iint_{f(R_{ij})} g$$

where we used the additive property of integrals over disjoint regions.

By the mean-value theorem for integrals, there is a point $f(u_{ij}^*, v_{ij}^*) \in f(R_{ij})$ such that $\iint_{f(R_{ij})} g = g(f(u_{ij}^*, v_{ij}^*))\, V(f(R_{ij}))$

If R_{ij} is sufficiently small,

$$\frac{\| f(u,v) - f(u_{ij}^*, v_{ij}^*) - Df(u_{ij}^*, v_{ij}^*)(u - u_{ij}^*, v - v_{ij}^*) \|}{\| (u - u_{ij}^*, v - v_{ij}^*) \|} < \epsilon$$

for all $(u,v) \in R_{ij}$

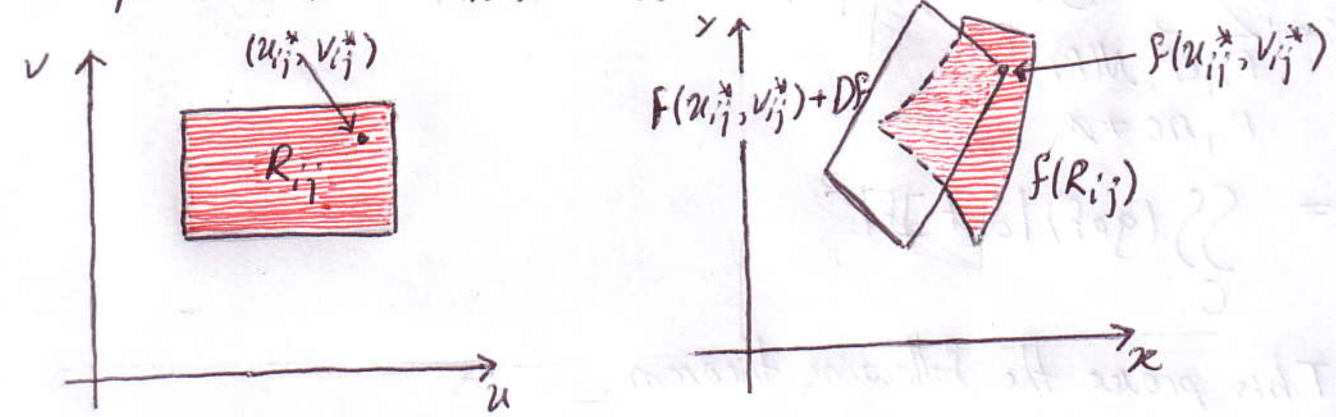
In particular,

$$f(u,v) \approx f(u_{ij}^*, v_{ij}^*) + Df(u_{ij}^*, v_{ij}^*)(u - u_{ij}^*, v - v_{ij}^*)$$

It is therefore reasonable to assume that

$$V(f(R_{ij})) \approx V(Df(u_{ij}^*, v_{ij}^*)(R_{ij})) = |\det Jf(u_{ij}^*, v_{ij}^*)|\, V(R_{ij}) =$$
$$= |\det Jf(u_{ij}^*, v_{ij}^*)|\, \Delta u_i \, \Delta v_j$$

as the picture below illustrates.



Thus

$$\iint_B g \approx \sum_{R_{ij} \cap C \neq \emptyset} \iint_{f(R_{ij})} g = \sum_{R_{ij} \cap C \neq \emptyset} g(f(u_{ij}^*, v_{ij}^*))\, V(f(R_{ij})) \approx$$

$$\approx \sum_{R_{ij} \cap C \neq \emptyset} g(f(u_{ij}^*, v_{ij}^*))\, |\det Jf(u_{ij}^*, v_{ij}^*)|\, \Delta u_i \, \Delta v_j$$

where the last expression is a Riemann sum of the integral $\iint_C (g \circ f) |\det Jf|$. Therefore, as the partition P gets finer, we see that

$$\iint_B g = \lim_{\delta \to 0^+} \sum_{\substack{R_{ij} \in P,\ N(P) < \delta \\ R_{ij} \cap C \neq \emptyset}} \iint_{f(R_{ij})} g =$$

$$= \lim_{\substack{\delta \to 0^+}} \sum_{\substack{R_{ij} \in P,\, N(P) < \delta \\ R_{ij} \cap C \neq \emptyset}} g(f(u_{ij}^*, v_{ij}^*))\, V(f(R_{ij})) =$$

$$= \lim_{\substack{\delta \to 0^+}} \sum_{\substack{R_{ij} \in P,\, N(P) < \delta \\ R_{ij} \cap C \neq \emptyset}} g(f(u_{ij}^*, v_{ij}^*))\, |\det Jf(u_{ij}^*, v_{ij}^*)|\, \Delta u_i \Delta v_j =$$

$$= \iint_C (g \circ f)\, |\det Jf|$$

This proves the following theorem.

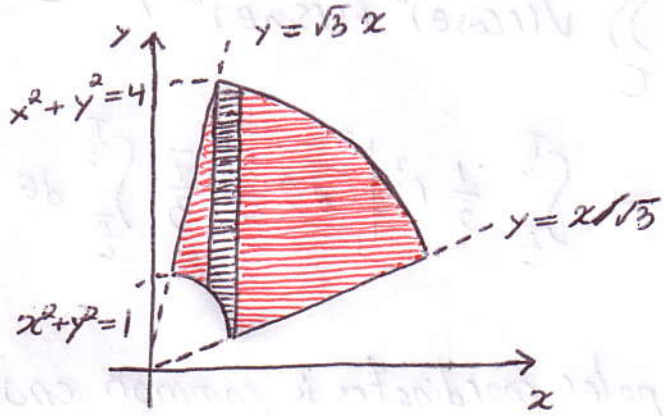
Thm (Change of variables formula for double integrals): Let B be a region in the xy-plane that is parametrized by $(x,y) = f(u,v)$ for $(u,v) \in C$, where f is differentiable on C with $\det Jf(u,v) \neq 0$ for all $(u,v) \in C$ and f is one-to-one on C. For any function g that is continuous on B,

$$\iint_B g = \iint_C (g \circ f)\, |\det Jf|$$

In the following examples, we illustrate how this theorem can be used.

Ex. Let B be the region in the first quadrant of the xy-plane between the lines $y = x/\sqrt{3}$ and $y = \sqrt{3}\, x$ outside the circle $x^2+y^2=1$ and inside the circle $x^2+y^2=4$. Evaluate $\iint_B \sqrt{x^2+y^2}$

Solution:



The direct approach is to write the double integral as the sum of three integrals over the y-simple regions shown above. But since B is simply described by the inequalities

$$1 \le r \le 2$$
$$\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$$

in polar coordinates, we will obtain a simpler integral by changing to polar variables r, θ.

Let $C = \{(r,\theta) : 1 \le r \le 2,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\}$. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(r,\theta) = (r\cos\theta, r\sin\theta)$. Then $B = f(C)$.

Furthermore, f is one-to-one on C (Why?).

Now, $Jf(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$ so $|\det Jf(r,\theta)| =$

$= |r\cos^2\theta + r\sin^2\theta| = r.$

By the change of variables theorem,

$$\iint_B \sqrt{x^2+y^2} = \iint_C \sqrt{(r\cos\theta)^2+(r\sin\theta)^2}\, r = \iint_C r^2 =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_1^2 r^2\, dr\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{3} r^3 \Big|_1^2 d\theta = \frac{7}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta = \frac{7\pi}{18}.$$

Changing variables to polar coordinates is common enough that we should write down a formula for this special case. In the example above we already saw that

$$|\det J(r\cos\theta, r\sin\theta)| = \left|\det \frac{\partial(r\cos\theta, r\sin\theta)}{\partial(r,\theta)}\right| = r$$

so if B is a region in the xy-plane that is the image of $C = \{(r,\theta) : f_1(\theta) \le r \le f_2(\theta),\ \theta_1 \le \theta \le \theta_2\}$, then

$$\iint_B g = \iint_C g\circ(r\cos\theta, r\sin\theta)\,|r| = \int_{\theta_1}^{\theta_2}\int_{f_1(\theta)}^{f_2(\theta)} r\, g(r\cos\theta, r\sin\theta)\, dr\, d\theta.$$

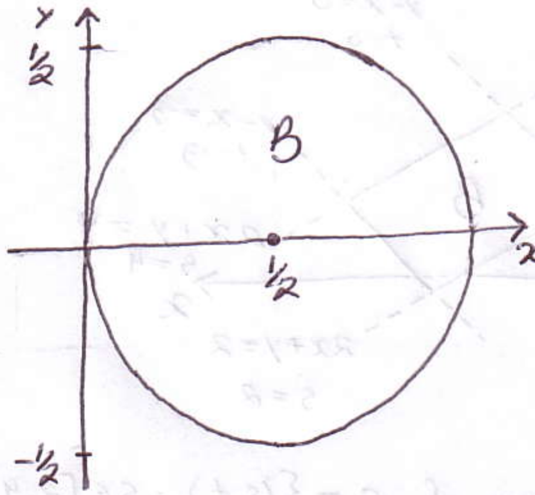
Ex. Evaluate $\iint_B \frac{1}{\sqrt{1-x^2-y^2}}$ where B is the disc of diameter 1 centered at $(1/2, 0)$

Solution: The region B is a disc of radius $r = \frac{1}{2}$ and center $(1/2, 0)$. Thus, $B = \{(x,y) : (x-1/2)^2 + y^2 \le 1/4\}$. In polar coordinates, this becomes

$$0 \le (r\cos\theta - 1/2)^2 + (r\sin\theta)^2 \le \frac{1}{4}$$

which simplifies to

$$0 \le r \le \cos\theta$$



In particular, setting $C = \{(r,\theta) : 0 \le r \le \cos\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$. we get that $B = f(C)$ where $f: C \to B$ is given by $f(r,\theta) = (r\cos\theta, r\sin\theta)$

By the change of variables theorem,

$$\iint_B \frac{1}{\sqrt{1-x^2-y^2}} = \iint_C \frac{r}{\sqrt{1-r^2}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} \frac{r}{\sqrt{1-r^2}}\, dr\, d\theta =$$

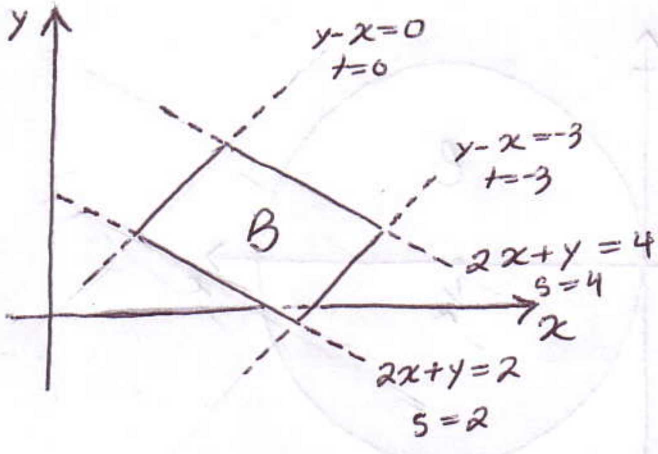
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\frac{1}{2} \cdot 2\sqrt{1-r^2}\Big|_0^{\frac{\pi}{2}} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin\theta)\, d\theta =$$

$$= (\theta + \cos\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi$$

Ex. Evaluate $\iint_B \frac{x-y}{(x+2y)^2}\, dy\, dx$ where B is the region bounded by the lines $2y+x=2$, $2y+x=4$, $y-x=-3$, and $y-x=0$

Solution: let $s = 2y+x$ and $t = y-x$, then the region is neatly described by

$$2 \le s \le 4$$
$$-3 \le t \le 0$$



In other words, B is the image of $C=\{(s,t): s\in[2,4],\ t\in[-3,0]\}$ under the transformation defined by these equations. We need to find det $\frac{\partial(x,y)}{\partial(s,t)}$. To do this, it appears that we must first solve for x and y in terms of s and t.

$$2y+x=s \qquad \begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}y\\ x\end{pmatrix}=\begin{pmatrix}s\\ t\end{pmatrix}$$
$$y-x=t$$

Therefore
$$\begin{pmatrix}y\\ x\end{pmatrix}=\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix}^{-1}\begin{pmatrix}s\\ t\end{pmatrix}=-\frac{1}{3}\begin{pmatrix}-1 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}s\\ t\end{pmatrix}$$

Thus $x=\frac{1}{3}s-\frac{2}{3}t$ and $y=\frac{1}{3}s+\frac{1}{3}t$.

In particular, $B=f(C)$ where $f(s,t)=\frac{1}{3}(s-2t,\ s+t)$.

Since f is linear $Jf(s,t)=\frac{1}{3}\begin{pmatrix}1 & -2\\ 1 & 1\end{pmatrix}$ and $\det Jf(s,t)=\frac{1}{3}$

By the change of variables theorem,

$$\iint_B \frac{x-y}{(x+2y)^2}=\iint_C \frac{-t}{s^2}\cdot\frac{1}{3}=\int_{-3}^{0}\int_{2}^{4}\frac{-t}{3s^2}\,ds\,dt=$$
$$=\frac{1}{3}\int_{-3}^{0} t\cdot\frac{1}{s}\Big|_{2}^{4}dt=-\frac{1}{12}\int_{-3}^{0}t\,dt=-\frac{1}{24}t^2\Big|_{-3}^{0}=\frac{3}{8}$$

In the previous example, we first inverted the transformation relating x and y to s and t, and then we calculated $\det \frac{\partial(x,y)}{\partial(s,t)}$ directly.

There is an alternative approach.

Suppose that x and y are related implicitly to s and t by the vector equation

$$(s,t) = k(x,y) \qquad (1)$$

where k is given. Now let f be the (unknown) function that gives x and y explicitly in terms of s and t. Then

$$(x,y) = f(s,t) \qquad (2)$$

Substituting (2) into (1), we get

$$(s,t) = k(f(s,t)) \qquad (3)$$

By the chain rule,

$$J(s,t) = Jk(f(s,t))\,Jf(s,t) = Jk(x,y)\,Jf(s,t)$$

or

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Jk(x,y)\,Jf(s,t) \qquad (4)$$

Thus the Jacobian matrices are inverses of one another

It follows that $1 = \det I_2 = \det(Jk(x,y)\,Jf(s,t)) =$
$= \det Jk(x,y) \det Jf(s,t)$.

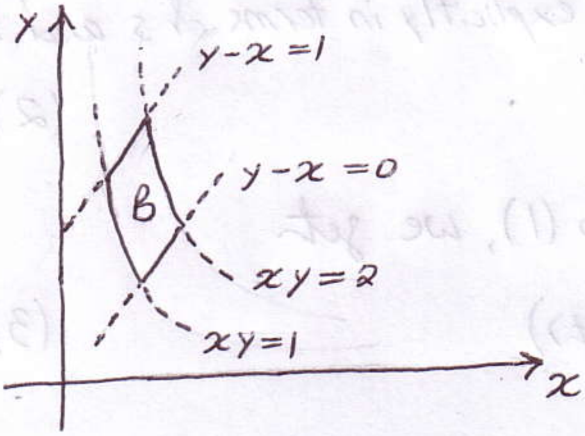
In particular,

$$\det Jf(s,t) = \frac{1}{\det JK(x,y)}. \qquad (\text{why?})$$

We can put the above formula to good use in many instances.

Ex. Evaluate $\iint_B (x^2-y^2)$ where B is the region in the first quadrant of the xy-plane bounded by the lines $y-x=0$, $y-x=1$ and the curves $xy=1$ and $xy=2$

Solution:



The region in question is shown above. We change to st-coordinates, where $s = y-x$ and $t = xy$. Then B is the image of $C = \{(s,t): s \in [0,1],\ t \in [1,2]\}$ under the transformation given implicitly by these equations. Proceeding directly to find x and y in terms of s and t becomes an arduous algebra problem so we attempt to use our new formula.

$$\det \frac{\partial(x,y)}{\partial(s,t)} = \frac{1}{\det \frac{\partial(s,t)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} -1 & 1 \\ y & x \end{vmatrix}} = \frac{1}{-x-y}$$

Thus we have

$$\iint_B (x^2-y^2) = \iint_C (x^2-y^2)\cdot\left|\frac{1}{-x-y}\right| ds\,dt =$$

$$= \iint_C \frac{x^2-y^2}{x+y}\,ds\,dt = \iint_C -(y-x)\,ds\,dt = \iint_C -s =$$

$$= -\int_0^1\int_1^2 s\,dt\,ds = -\int_0^1 s\,ds = -\frac{1}{2}s^2\Big|_0^1 = -\frac{1}{2}.$$

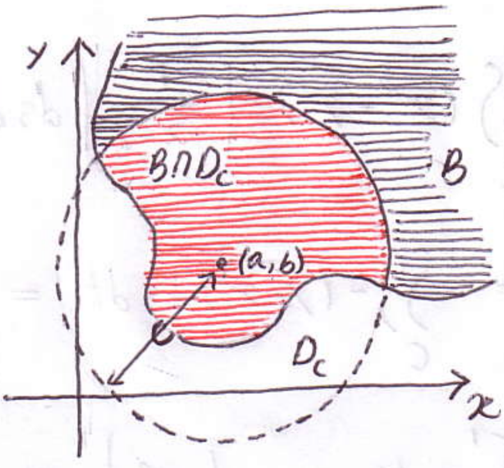
As a final example, we show how changing variables can be used to evaluate a double integral in which the region of integration is unbounded. Let $f(x,y)$ be a nonnegative continuous function on an unbounded region B in the xy-plane, and let (a,b) be any point in B. Then for any positive number c, f is integrable on the portion of B that lies in the disc D_c of radius c centered at (a,b). That is, we can calculate $\iint_{B\cap D_c} f$ for each choice of c.

As we take c to be larger and larger, these integrals either approach a number or they do not. If they do, we say that $\iint_B f$ converges and write

$$\iint_B f = \lim_{c\to\infty} \iint_{B\cap D_c} f$$

otherwise, we say that it diverges.

See figure below.



As D_c expands, it engulfs the entire region B.

Ex. Evaluate $\iint_B e^{-x^2-y^2}$ where B is the first quadrant of the xy-plane.

Solution: Let D_c be a disc of radius c centered at $(0,0)$. Then $D_c \cap B$ is just the sector given in polar coordinates by $0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le c$. Converting the integral to polar coordinates, we have

$$\iint_{B \cap D_c} e^{-x^2-y^2} = \int_0^{\frac{\pi}{2}} \int_0^c e^{-r^2} r \, dr \, d\theta = \int_0^{\frac{\pi}{2}} -\frac{1}{2} e^{-r^2} \Big|_0^c d\theta =$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1-e^{-c^2}) \, d\theta = \frac{1}{4}(1-e^{-c^2})\pi$$

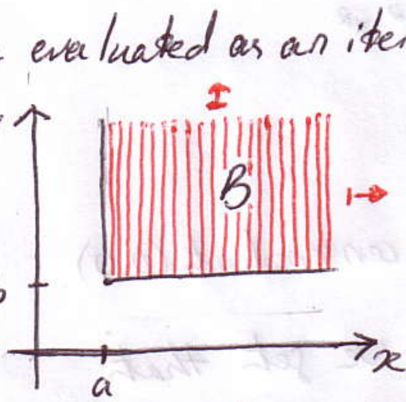
Thus

$$\iint_B e^{-x^2-y^2} = \lim_{c\to\infty} \iint_{B \cap D_c} e^{-x^2-y^2} = \lim_{c\to\infty} \frac{1}{4}(1-e^{-c^2})\pi = \frac{1}{4}\pi$$

If B is, for instance, the quarter-plane described by $a \le x$ and $b \le y$ and $\iint_B f$ converges, then it follows that $\int_b^\infty f(x,y)\,dy$ converges for each choice of x in the interval $[a,\infty)$ and also that

$$\int_a^\infty \left(\int_b^\infty f(x,y)\,dy \right) dx$$

converges. In other words, a double integral over a region of this special type can be evaluated as an iterated improper integral. (Why?)



The quarter-plane
$a \le x,\ b \le y$

Ex. Evaluate $\iint_B \frac{1}{x^2y^3}$ where B is the region described by $1 \le x$ and $5 \le y$

Solution: If the integral converges, then it is equal to the iterated integral

$$\int_1^\infty \int_5^\infty \frac{1}{x^2y^3}\,dy\,dx$$

Now $\int_5^\infty \frac{1}{x^2y^3}\,dy = \lim_{b\to\infty} \int_5^b \frac{1}{x^2y^3}\,dy = \lim_{b\to\infty} \frac{1}{x^2}\cdot\frac{-1}{2y^2}\Big|_5^b = \frac{1}{50x^2}$

for each $x \ge 1$.

Thus

$$\int_1^\infty \int_5^\infty \frac{1}{x^2y^3}\,dy\,dx = \int_1^\infty \frac{1}{50}\frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{50}\frac{1}{x^2}\,dx = \frac{1}{50}$$

Ex. Evaluate $\int_{-\infty}^{\infty} e^{-x^2}\,dx$

Solution: Let $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. Then $I = \int_{-\infty}^{\infty} e^{-y^2}\,dy$.

Therefore $I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy =$

$$= \iint_{\mathbb{R}^2} e^{-x^2-y^2} = \lim_{c\to\infty}\iint_{D_c} e^{-x^2-y^2}$$

where D_c is a disc of radius c centered at $(0,0)$.

By switching to polar coordinates, we get that

$$\iint_{D_c} e^{-x^2-y^2} = \int_0^{2\pi}\int_0^c re^{-r^2}\,dr\,d\theta = -\frac{1}{2}\int_0^{2\pi} e^{-r^2}\Big|_0^c d\theta =$$

$$= -\frac{1}{2}\int_0^{2\pi}(-1+e^{-c^2})\,d\theta = \frac{1-e^{-c^2}}{2}\,2\pi$$

Thus $\lim_{c\to\infty}\iint_{D_c} e^{-x^2-y^2} = \lim_{c\to\infty}\frac{1-e^{-c^2}}{2}\,2\pi = \pi$.

It follows that $I = \sqrt{I^2} = \sqrt{\pi}$.

Change of Variables for triple integrals

Thm: Let $g: \mathcal{U} \subset \mathbb{R}^3 \to \mathbb{R}$ be continuous and suppose that $B \subseteq \mathcal{U}$ is closed and bounded. If $C \subset \mathbb{R}^3$ is a closed and bounded set and $f: C \to B$ is a smooth one-to-one function that satisfies $f(C) = B$, then

$$\iiint_B g = \iiint_C (g \circ f) |\det Jf|$$

Proof: Although the proof is essentially the same as in the two variable case, it would be useful to go through the argument yet again.

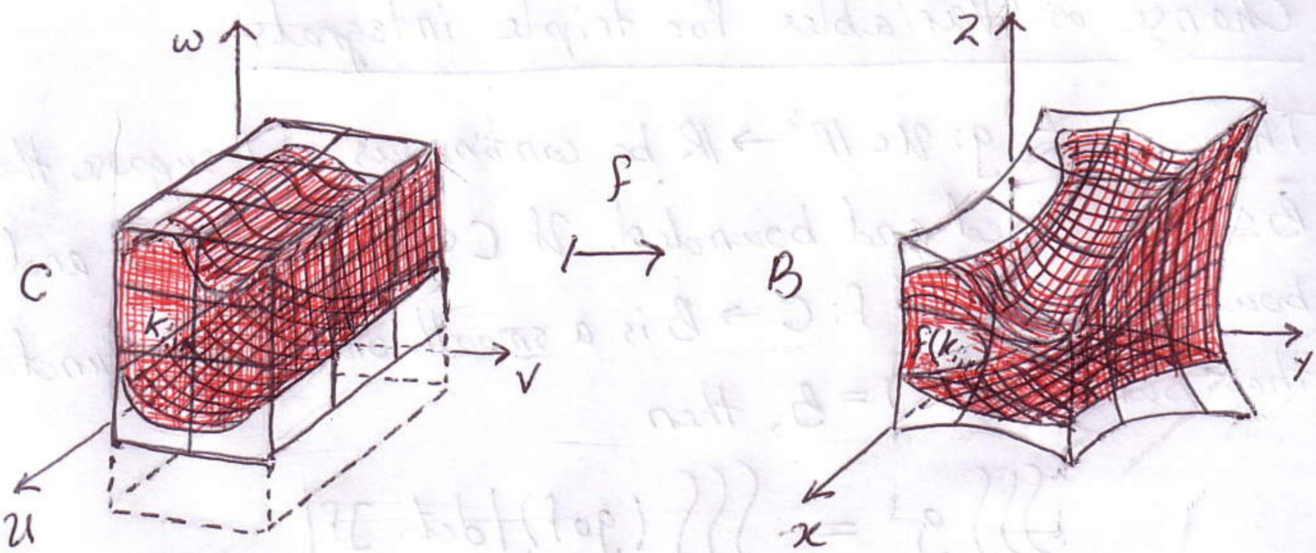
Enclose the solid region C in a box and partition this box into a fine mesh of small sub-boxes K_{ijk}. Then

$$B = f(C) \approx f\Big(\bigcup_{K_{ijk} \cap C \neq \emptyset} K_{ijk}\Big) = \bigcup_{K_{ijk} \cap C \neq \emptyset} f(K_{ijk}).$$

Therefore,

$$\iiint_B g = \iiint_{f(C)} g \approx \iiint_{\substack{\cup f(K_{ijk}) \\ K_{ijk} \cap C \neq \emptyset}} g = \sum_{K_{ijk} \cap C \neq \emptyset} \iiint_{f(K_{ijk})} g$$

See the picture below for illustration.



By the mean value theorem for integrals, there exists a point $f(u^*_{ijk}, v^*_{ijk}, w^*_{ijk}) \in f(K_{ijk})$ such that

$$\iiint_{f(K_{ijk})} g = g\left(f(u^*_{ijk}, v^*_{ijk}, w^*_{ijk})\right) V\left(f(K_{ijk})\right)$$

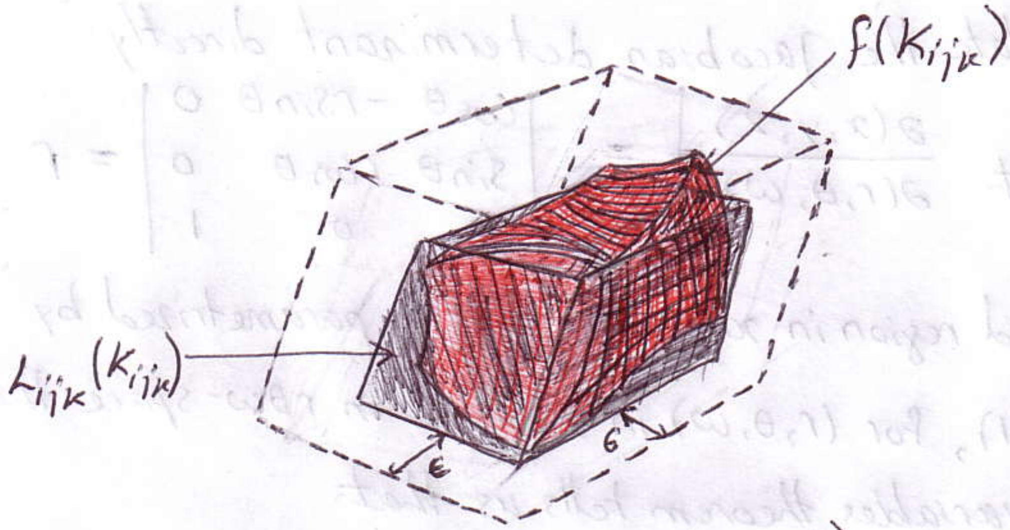
where $V(f(K_{ijk}))$ is the volume of the solid region $f(K_{ijk})$.

Let $L_{ijk}(u,v,w) = f(u^*_{ijk}, v^*_{ijk}, w^*_{ijk}) + Df(u^*_{ijk}, v^*_{ijk}, w^*_{ijk})(\Delta u^*_i, \Delta v^*_j, \Delta w^*_k)$ where $(u-u^*_{ijk}, v-v^*_{ijk}, w-w^*_{ijk}) = (\Delta u^*_i, \Delta v^*_j, \Delta w^*_k)$, be the linear approximation of f at the point $(u^*_{ijk}, v^*_{ijk}, w^*_{ijk})$. Then, for K_{ijk} sufficiently small (i.e $N(K_{ijk}) < \delta_{ijk}$), the distance $\| f(u,v,w) - L_{ijk}(u,v,w)\|$ is less than ϵ whenever $(u,v,w) \in K_{ijk}$. Therefore,

$$V(f(K_{ijk})) \approx V(L_{ijk}(K_{ijk})) = \left|\det Jf(u^*_{ijk}, v^*_{ijk}, w^*_{ijk})\right| V(K_{ijk})$$

$$= \left| \det Jf(u^*_{ijk}, v^*_{ijk}, w^*_{ijk}) \right| \Delta u_i \Delta v_j \Delta w_k$$

See illustration below.



$$V\left(L_{ijk}(K_{ijk})\right) \approx V(f(K_{ijk}))$$

because $f(K_{ijk})$ is contained within a parallelopiped similar to $L_{ijk}(K_{ijk})$ with dimensions bigger by ϵ.

It follows that

$$\iiint_B g = \lim_{\delta \to 0^+} \sum_{K_{ijk} \cap C \neq \emptyset} g(f(u^*_{ijk}, v^*_{ijk}, w^*_{ijk})) \left|\det Jf(u^*_{ijk}, v^*_{ijk}, w^*_{ijk})\right|$$

$$\Delta u_i \Delta v_j \Delta w_k = \iiint_C (g \circ f) \left|\det Jf\right|$$

which establishes the desired result.

We illustrate the use of this formula for cylindrical coordinates. Since the transformation from cylindrical to rectangular coordinates is given by

$$x = r\cos\theta \qquad\qquad (1)$$
$$y = r\sin\theta$$
$$z = \omega$$

we can calculate the Jacobian determinant directly:

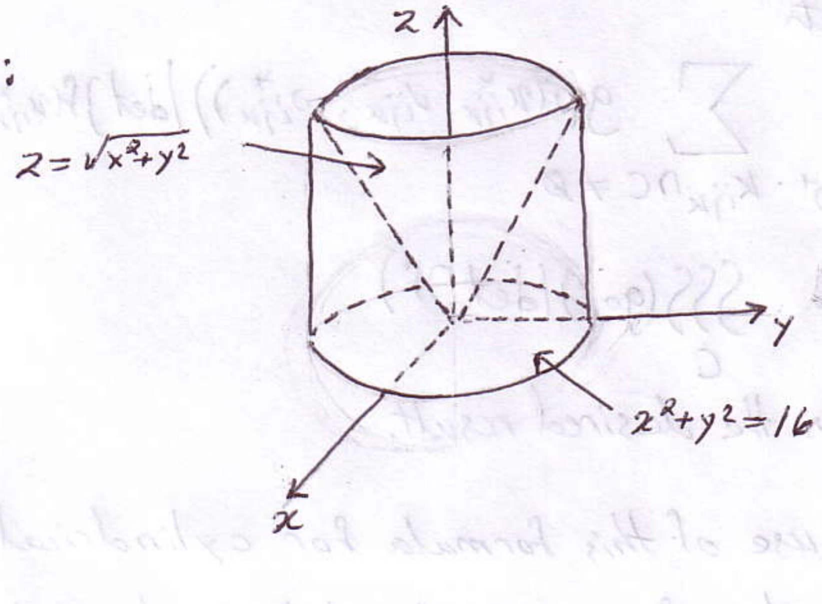
$$\left|\det \frac{\partial(x,y,z)}{\partial(r,\theta,\omega)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

If B is a solid region in xyz-space that is parametrized by the equations (1), for (r,θ,ω) in a solid C in $r\theta\omega$-space then the change of variables theorem tells us that

$$\iiint_B g = \iiint_C g(r\cos\theta, r\sin\theta, \omega)\, r\, dr\, d\theta\, d\omega.$$

Ex. Evaluate $\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_0^{\sqrt{x^2+y^2}} z + x \, dz\, dy\, dx.$

Solution:



The region of integration is inside the cylinder $x^2+y^2 \leq 16$ above the plane $z=0$ and below the cone $z=\sqrt{x^2+y^2}$. In cylindrical coordinates it has the simple description

$$0 \leq r \leq 4$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq \omega \leq r$$

Thus the integral is

$$\int_0^4 \int_0^{2\pi} \int_0^r (\omega + r\cos\theta) r \, d\omega \, d\theta \, dr = \int_0^4 \int_0^{2\pi} \left(\frac{1}{2} + \cos\theta\right) r^3 d\theta \, dr$$

$$= \int_0^4 \pi r^3 dr = \frac{\pi}{4} r^4 \Big|_0^4 = 64\pi.$$

Next, we obtain a change of variables formula for spherical coordinates. The transformation from spherical coordinates to rectangular is given by the three equations

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho \cos\varphi \end{aligned} \qquad (2)$$

so the Jacobian determinant is

$$\det \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \det \begin{pmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{pmatrix}$$

$$= -\rho^2 \sin\varphi \cos^2\varphi - \rho^2 \sin^3\varphi = -\rho^2 \sin\varphi.$$

Thus, if B is a solid region in xyz-space that is parametrized by (2), for (ρ, θ, φ) in a solid region C of $\rho\theta\varphi$-space, then, by the change of variables theorem,

$$\iiint_B g = \iiint_C g(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi.$$

Ex. Evaluate $I = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} k\sqrt{x^2+y^2+z^2}\, dx\,dy\,dz.$

Solution: The solid on which we integrate is the ball described by $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$, and $0 \le \varphi \le \pi$. Thus

$$I = \int_0^1\int_0^\pi\int_0^{2\pi} k\rho\cdot\rho^2\sin\varphi\, d\theta\, d\varphi\, d\rho = 2\pi k\int_0^1\int_0^\pi \rho^3\sin\varphi\, d\varphi\, d\rho$$

$$= 2\pi k\int_0^1 \rho^3(-\cos\varphi)\Big|_0^\pi d\rho = 4\pi k\cdot\frac{1}{4}\rho^4\Big|_0^1 = \pi k.$$

This integration is immensely simpler than carrying out the calculations using rectangular coordinates!

Ex (Gilligan's Island). Recall that the expression for the mass of blood with spill radius R is given by

$$M(R) = \int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{-\sqrt{R^2-x^2-y^2}}^{0} 1060\,\frac{(\sqrt{x^2+y^2+z^2}-R)^{16}}{R^{16}}\, dz\,dy\,dx$$

The solid over which the integral is taken is the half ball described by $0 \le \rho \le R$, $0 \le \theta \le 2\pi$, and $\frac{\pi}{2} \le \varphi \le \pi$.

Therefore, by the change of variable theorem,

$$M(R) = \int_0^R\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} 1060\,\frac{(\rho-R)^{16}}{R^{16}}\rho^2\sin\varphi\, d\theta\, d\varphi\, d\rho$$

$$= \frac{2\pi \cdot 1060}{R^{16}} \int_0^R \int_{\frac{\pi}{2}}^{\pi} (p-R)^{16} p^2 \sin\varphi \, d\varphi \, dp =$$

$$= \frac{2120\pi}{R^{16}} \int_0^R p^2 (p-R)^{16} (-\cos\varphi) \Big|_{\frac{\pi}{2}}^{\pi} dp =$$

$$= \frac{2120\pi}{R^{16}} \int_0^R p^2 (p-R)^{16} dp$$

Let $u = p - R$ then $p^2 = (u+R)^2$ and $du = dp$.

Therefore,

$$M(R) = \frac{2120\pi}{R^{16}} \int_{-R}^{0} (u+R)^2 u^{16} du =$$

$$= \frac{2120\pi}{R^{16}} \int_{-R}^{0} (u^{18} + 2u^{17}\cancel{R} + u^{16}R^2) du =$$

$$= \frac{2120\pi}{R^{16}} \left(\frac{u^{19}}{19} + 2\frac{u^{18}R}{18} + \frac{u^{17}R^2}{17} \right) \Big|_{-R}^{0} =$$

$$= \frac{2120\pi}{R^{16}} \left(\frac{R^{19}}{19} + \frac{R^{19}}{17} - \frac{R^{19}}{9} \right) = \frac{2120\pi}{2907} R^3$$

Recall that the pool of blood that Gilligan observed has radius $R = 2.9\,m$. Therefore the mass of blood is $M(2.9) = \frac{2120}{2907}\pi(2.9)^3 \approx 55.8 > 10.5$, which translates to a care-free Sunday night!

In some cases, the change of variables may be given implicitly by a set of equations. For instance, B might be the solid bounded by the six planes $x+y+z=0$, $x+y+z=7$, $2x+y+3z=0$, $2x+y+3z=2$, $-x+y-z=1$, and $-x+y-z=8$. In this case it would be reasonable to let $s = x+y+z$, $t = 2x+y+3z$, and

$u = -x + y - z$ so that B is the image of $C = [0,7] \times [0,2] \times [1,8]$ under the function of s, t, and u defined implicitly by these equations. Here we can solve for x, y, and z in terms of s, t, and u to find this function, but this may be impractical or impossible in other situations. Nevertheless, we can sometimes still make a change of variables by using the three-variable analogue of the property established on p. 12.

Suppose that k is a given differentiable function and that f is another differentiable function that satisfies

$$f(k(s,t,u)) = (s,t,u)$$

with

$$(s,t,u) = k(x,y,z)$$

and

$$(x,y,z) = f(s,t,u)$$

it follows that

$$\det\left(\frac{\partial(x,y,z)}{\partial(s,t,u)}\right) = \frac{1}{\det(\partial(s,t,u)/\partial(x,y,z))}.$$

We illustrate how this is used in the next example.

Ex. Evaluate $\iiint_B x^2$ where B is the solid bounded by the six planes $x+y=1$, $x+y=2$, $2x+y+z=-1$, $2x+y+z=1$, $x-2y=0$, and $x-2y=1$

Solution: A reasonable change of coordinates is to let

$$s = x + y$$
$$t = 2x + y + z$$
$$u = x - 2y$$

so that B is the image of $C = \{(s,t,u) : s \in [1,2], t \in [-1,1], u \in [0,1]\}$ under the transformation whose inverse is given by these formulas. Since this is a linear transformation, its inverse is also linear, and we could readily calculate it. Instead, observe that

$$\det \frac{\partial(s,t,u)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & -2 & 0 \end{vmatrix} = 3$$

Therefore,

$$\det \frac{\partial(x,y,z)}{\partial(s,t,u)} = \frac{1}{3}$$

To find x in terms of s, t, and u, we may apply Cramer's rule:

$$\frac{\begin{vmatrix} s & 1 & 0 \\ t & 1 & 1 \\ u & -2 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & -2 & 0 \end{vmatrix}} = x$$

In particular, $x = \frac{2}{3}s + \frac{1}{3}u$.

Then, by the change of variables theorem,

$$\iiint_B x^2 = \int_1^2 \int_0^1 \int_{-1}^1 \left(\frac{2}{3}s + \frac{1}{3}u\right)^2 \frac{1}{3}\, dt\, du\, ds =$$

$$= \frac{1}{3} \int_1^2 \int_0^1 \left(\frac{2}{3}s + \frac{1}{3}u\right)^2 t \Big|_{-1}^{1} du\, ds =$$

$$= \frac{2}{3} \int_1^2 \left(\frac{2}{3}s + \frac{1}{3}u\right)^3 \Big|_0^1 ds =$$

$$= \left[\frac{1}{4}\left(\frac{2}{3}s + \frac{1}{3}\right)^4 - \frac{1}{2}s^4\right]\Big|_1^2 = \frac{76}{81}$$