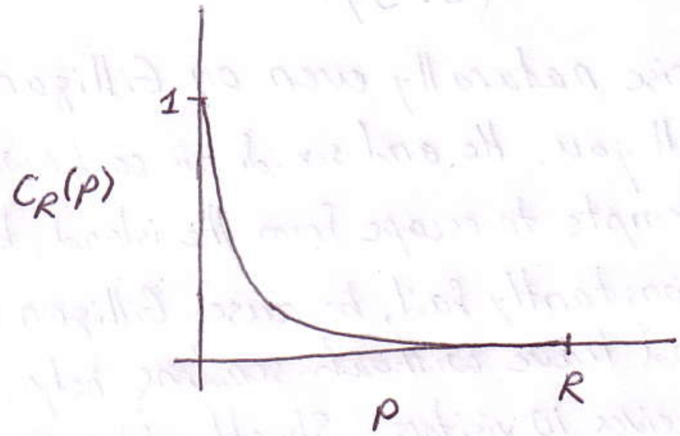
(5.5)

Triple integrals arise naturally even on Gilligan's island as he can, no doubt, tell you. He and six of his castaway friends have made numerous attempts to escape from the island, but to no avail. Their escape plans constantly fail, because Gilligan goofs up or visitors to the island leave without sending help.

One time, Gilligan receives 10 visitors. Shortly after their departure on a makeshift raft, he looks through his binoculars and sees a huge pool of blood not far from the shore. He estimates the pool of blood to be 5.8 meters in diameter, which reminds him that he has forgotten to feed his pet sharks yet again. This unfortunate accident could not have happened at a more inappropriate time; it is Sunday evening - a time to watch "60 minutes" and unwind. Must Gilligan pay for his little mistake and join a search party for survivors? From past experience, Gilligan knows that the density of blood is 1060 kg/m^3 and that the average person has a total of 5 kg of blood in his body. Gilligan estimates that blood spills into sea water at the same rate in every direction, so that the blood is contained in a hemisphere of radius $R = \frac{5.8}{2} = 2.9$ m. He also believes that blood concentration in sea water falls rapidly from 1 to 0 and is given by the function

$$C_R(\rho) = \begin{cases} 0 & \text{if } \rho > R \\ \dfrac{(\rho - R)^{16}}{R^{16}} & \text{if } \rho \le R \end{cases}$$

where ρ is the distance from the origin of the spill.



blood concentration as
a function of distance.

If Gilligan can somehow calculate the total mass of blood contained within the hemisphere of radius R, $M(R)$, he will not need to worry about being obliged to join a rescue operation whenever

$$M(R) \geq 5 \cdot n \text{ kg}$$

where n is the number of visitors to the island.

Gilligan can calculate the mass of an object whose density is constant throughout by multiplying the density by the object's volume. However, if the density of an object varies from point to point then some other procedure for finding the mass must be employed. A reasonable approach is first to assume that the densities at points nearby to one another are nearly the same. Then, by subdividing the object into many small portions on each of which the density is practically constant, Gilligan may approximate the mass by adding up the products of these densities and volumes.

Let S be a solid region in $\mathbb{R}^3$ that is bounded by a piecewise smooth boundary surface (i.e. a surface that can be decomposed into finitely many pieces, each of which is the graph of a function of two variables with continuous partials over a simple region.

in one of the coordinate planes). Let f be a real-valued function defined on S. If f is nonnegative, we can think of $f(x,y,z)$ as the density of the solid at the point (x,y,z). Surround S with a box $K = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, partition each interval as

$$a_1 = x_0 < x_1 < \ldots < x_{n_1} = b_1$$
$$a_2 = y_0 < y_1 < \ldots < y_{n_2} = b_2$$
$$a_3 = z_0 < z_1 < \ldots < z_{n_3} = b_3$$

and let $K_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ for $1 \le i \le n_1$, $1 \le j \le n_2$, $1 \le k \le n_3$. Choose a point $(x_{ijk}, y_{ijk}, z_{ijk}) \in K_{ijk}$ and form the sum

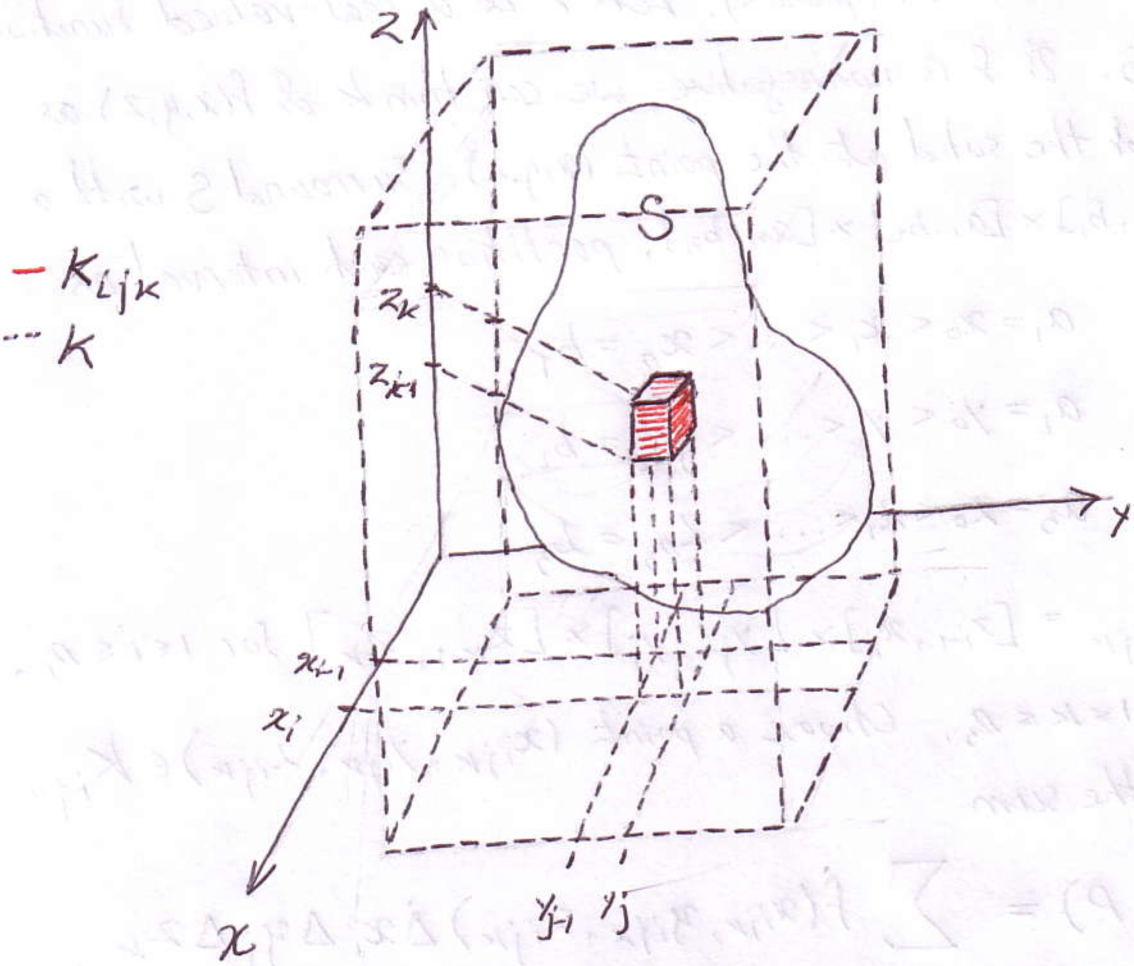
$$I(f, P) = \sum_{(x_{ijk}, y_{ijk}, z_{ijk}) \in S} f(x_{ijk}, y_{ijk}, z_{ijk}) \Delta x_i \Delta y_j \Delta z_k$$

where, as usual, $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, $\Delta z_k = z_k - z_{k-1}$. Imagine carrying out this process for all partitions P of norm less than some small positive number δ. [The norm $N(P)$ of a partition P is the maximum of the lengths of the diagonals of the rectangular boxes of P]

Then

$$\lim_{\delta \to 0^+} \sum_{\substack{(x_{ijk}, y_{ijk}, z_{ijk}) \in S \\ N(P) < \delta}} f(x_{ijk}, y_{ijk}, z_{ijk}) \Delta x_i \Delta y_j \Delta z_k = \iiint_S f$$

See picture below

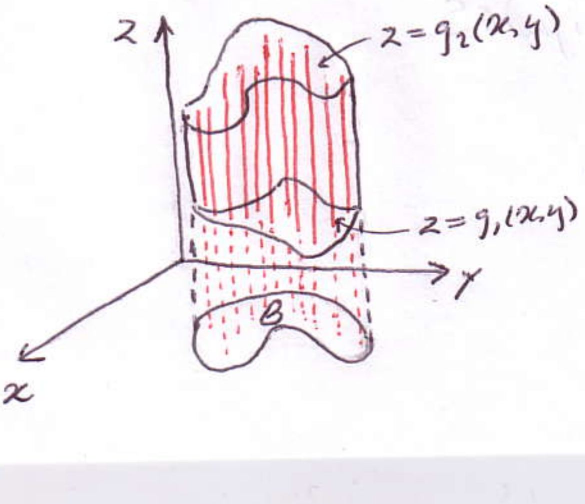


Just like with double integrals, triple integrals can often be evaluated as iterated integrals if the solid S is of a simple form.

Def: A solid $S \subset \mathbb{R}^3$ is z-simple if it is of the form

$$S = \{(x,y,z) : (x,y) \in B,\ g_1(x,y) \leq z \leq g_2(x,y)\}$$

for some functions g_1 and g_2 defined on a region B in the xy-plane.



A z-simple region is characterized by the fact that no vertical line emanating from the interior of B will intersect the surface enclosing the solid S, ∂S, more than two times.

Def: A solid $S \subset \mathbb{R}^3$ is y-simple if it is of the form

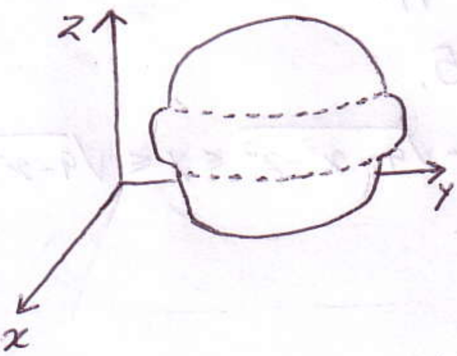
$$S = \{(x,y,z) : (x,z) \in B,\ h_1(x,z) \le y \le h_2(x,z)\}$$

for some functions h_1 and h_2 defined on a region B in the xz-plane.

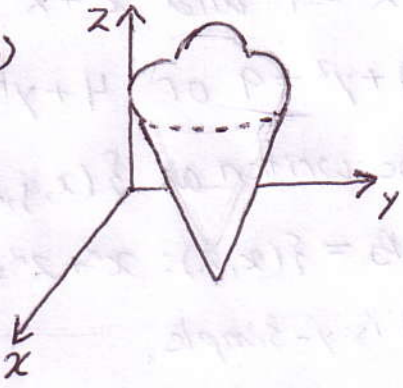
An x-simple region is defined similarly.

Ex. Which of the following objects are simple with respect to x, y, or z?

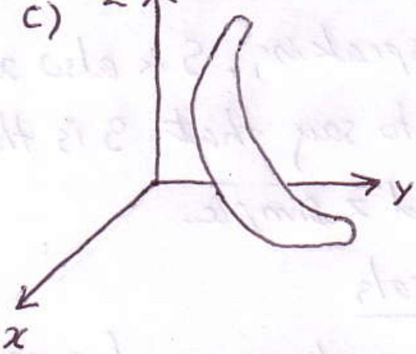
a)



b)



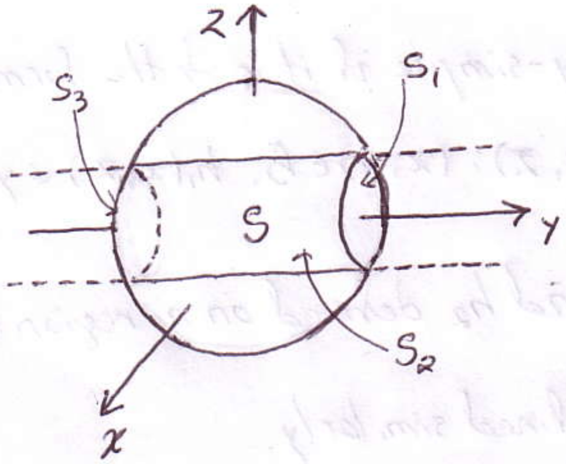
c)



Solution: (a) is x- and y-simple but not z-simple. (b) is simple (i.e. x, y, and z-simple), (c) is x- and y-simple but not z-simple.

Ex. Let S be the intersection of the ball $x^2+y^2+z^2 \leq 9$ and the (filled) cylinder $x^2+z^2 \leq 4$. Is S a simple solid?

Solution:



The boundary of S, ∂S, is the intersection of the surfaces $x^2+y^2+z^2=9$ and $x^2+z^2=4$, which happens when $(x^2+z^2)+y^2=9$ or $4+y^2=9 \Rightarrow y^2=5$.

S can be written as $\{(x,y,z): (x,z)\in B, -\sqrt{9-x^2-z^2} \leq y \leq \sqrt{9-x^2-z^2}\}$ where $B=\{(x,z): x^2+z^2 \leq 4\}$.

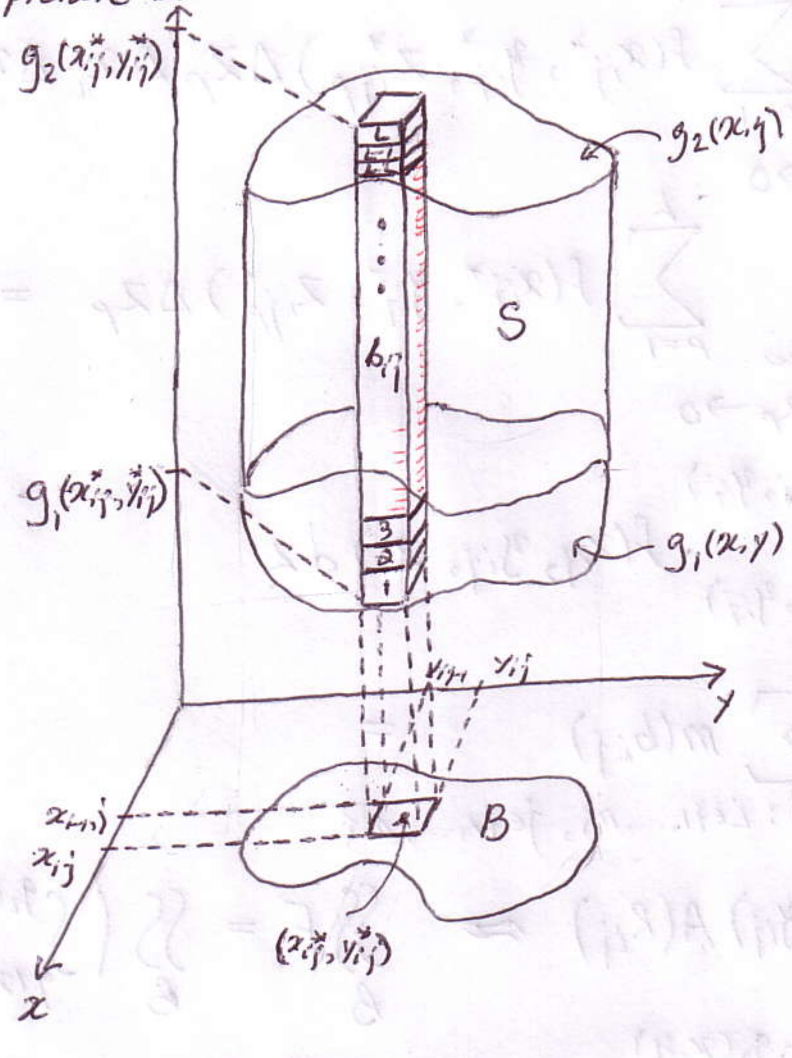
Thus S is y-simple.

Remark: Technically speaking, S is also x-and-z-simple. However, it is simpler to say that S is the union of S_1, S_2, and S_3, which are x, y, and z simple.

<u>Evaluating triple integrals</u>

Let $f: \mathcal{U} \subset \mathbb{R}^3 \to \mathbb{R}$ be continuous and suppose $S \subseteq \mathcal{U}$ is a z-simple solid. Then $S=\{(x,y,z): (x,y)\in B, g_1(x,y) \leq z \leq g_2(x,y)\}$.

To compute $\iiint_S f$, we may resort to Cavalieri's principle like we were doing on many occasions before. Unfortunately the graph of $f: \mathcal{U} \subset \mathbb{R}^3 \to \mathbb{R}$ is a three-dimensional solid in a four-dimensional space and its slices are not easily visualized. To assist your intuition, think of f as a density function and $\iiint_S f$ as the mass of the solid S. For the subsequent discussion refer to the picture below.



Partition the region B with nm rectangles R_{ij}. For each rectangle R_{ij} in the partition, choose a point $(x_{ij}^*, y_{ij}^*) \in R_{ij} \cap B$ and construct a rectangular beam b_{ij} in S of length $g_2(x_{ij}^*, y_{ij}^*) - g_1(x_{ij}^*, y_{ij}^*)$. Let $M(b_{ij})$ be the mass of the beam over the rectangle R_{ij}. If R_{ij} is small and the beam b_{ij} is sliced into h tiny boxes K_{ijk}, $k \in \{1, \dots, h\}$, then $M(b_{ij}) \approx \sum_{p=1}^{h} f(x_{ij}^*, y_{ij}^*, z_{ijp}^*) \Delta z_p \Delta x_i \Delta y_j$

where $(x_{ij}^*, y_{ij}^*, z_{ijp}) \in K_{ijp}$ for $p \in \{1, \ldots, L\}$ and where $g_1(x_{ij}, y_{ij}) = z_{ij0} < z_{ij1} < \ldots < z_{ijL} = g_2(x_{ij}, y_{ij})$ is a partition of the interval $[g_1(x_{ij}, y_{ij}), g_2(x_{ij}, y_{ij})]$.

Thus

$$M(b_{ij}) = \lim_{\substack{L \to \infty \\ \max \Delta z_p \to 0}} \sum_{p=1}^{L} f(x_{ij}^*, y_{ij}^*, z_{ijp}^*) \Delta z_p \Delta x_i \Delta y_j =$$

$$= \Delta x_i \Delta y_j \lim_{\substack{L \to \infty \\ \max \Delta z_p \to 0}} \sum_{p=1}^{L} f(x_{ij}^*, y_{ij}^*, z_{ijp}^*) \Delta z_p =$$

$$= A(R_{ij}) \int_{g_1(x_{ij}, y_{ij})}^{g_2(x_{ij}, y_{ij})} f(x_{ij}, y_{ij}, z)\, dz$$

Since

$$M(S) \approx \sum_{i\ j:\ i \in \{1,\ldots,n\},\ j \in \{1,\ldots,m\}} m(b_{ij}) =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} F(x_{ij}, y_{ij}) A(R_{ij}) \approx \iint_B F = \iint_B \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz \right),$$

where $F(x,y) = \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz$ and $A(R_{ij})$ is the area of the rectangle R_{ij}, it follows that

$$\iiint_S f = \iint_B \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz \right)$$

whenever S is z-simple. We state this conclusion in the following theorem:

Thm: If f is continuous on a solid $S \subset \mathbb{R}^3$ that is z-simple, then

$$\iiint_S f = \iint_B \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz \right) \qquad (1)$$

If S is y-simple, then

$$\iiint_S f = \iint_B \left(\int_{h_1(x,z)}^{h_2(x,z)} f(x,y,z)\,dy \right) \qquad (2)$$

and if S is x-simple, then

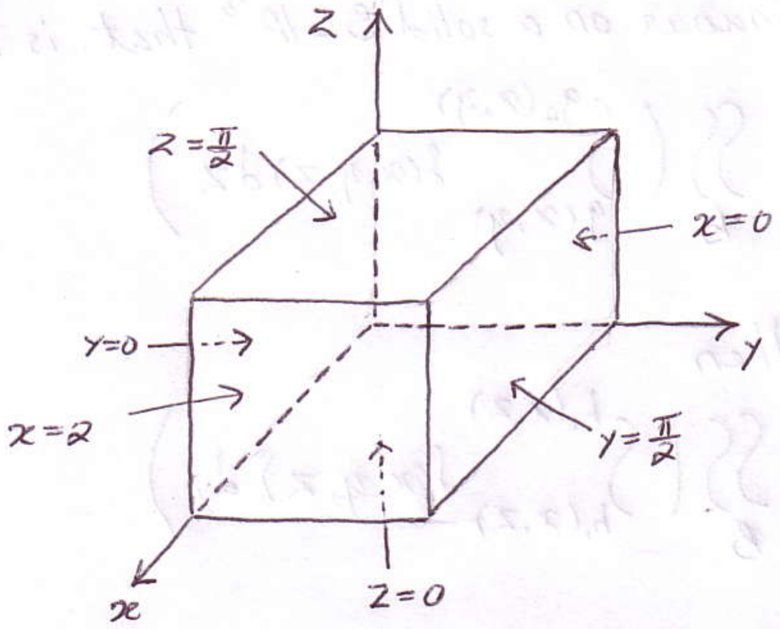
$$\iiint_S f = \iint_B \left(\int_{k_1(y,z)}^{k_2(y,z)} f(x,y,z)\,dx \right) \qquad (3)$$

Ex, Evaluate the integral $\iiint_S x \sin(y+z)$, where S is the solid bounded by the planes $x=0$, $x=2$, $y=0$, $y=\frac{\pi}{2}$, $z=0$, and $z=\frac{\pi}{2}$.

Solution: The solid S is just the box bounded by

$$0 \le x \le 2$$
$$0 \le y \le \frac{\pi}{2}$$
$$0 \le z \le \frac{\pi}{2}$$

Thus, S is x, y, and z simple. See picture below:



Treating S as z-simple, we have

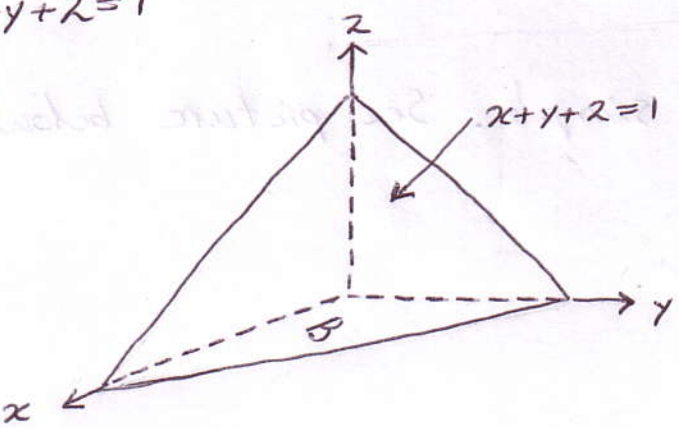
$$\iiint_S x\sin(y+z) = \iint_{[0,2]\times[0,\frac{\pi}{2}]} \left(\int_0^{\frac{\pi}{2}} x\sin(y+z)\,dz\right) =$$

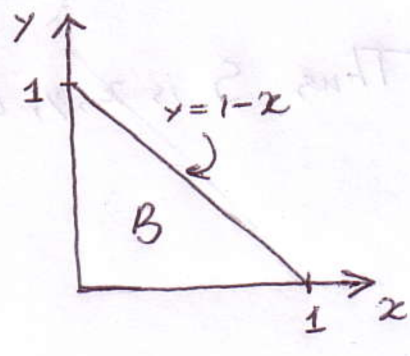
$$= \int_0^2\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} x\sin(y+z)\,dz\,dy\,dx = \int_0^2\int_0^{\frac{\pi}{2}} x(\sin y+\cos y)\,dy\,dx =$$

$$= \int_0^2 x(-\cos y+\sin y)\Big|_0^{\frac{\pi}{2}}dx = \int_0^2 2x\,dx = 4.$$

Ex. Evaluate $\iiint_S f$, where $f(x,y,z) = xy + yz$ and S is the "unit tetrahedron" bounded by the coordinate planes and the plane $x+y+z=1$

Solution:





The solid is evidently x-, y-, and z-simple. We choose to treat it as z-simple; it consists of the points (x, y, z) satisfying

$$0 \le z \le 1 - x - y \quad (x,y) \in B,$$

where B is the triangle in the xy-plane bounded by the coordinate axes and the line $x+y=1$. Treating B as y-simple, we can describe it by the inequalities

$$0 \le y \le 1 - x$$
$$0 \le x \le 1$$

Thus

$$\iiint_S (xy + yz) = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (xy + yz)\, dz\, dy\, dx =$$

$$= \int_0^1 \int_0^{1-x} \left(xy(1-x-y) + \tfrac{1}{2} y (1-x-y)^2 \right) dy\, dx =$$

$$= \int_0^1 \int_0^{1-x} \left(\tfrac{1}{2} y^3 - y^2 + \tfrac{1}{2} y - \tfrac{1}{2} x^2 y \right) dy\, dx =$$

$$= \int_0^1 \left(\tfrac{1}{8}(1-x)^4 - \tfrac{1}{3}(1-x)^3 + \tfrac{1}{4}(1-x)^2 - \tfrac{1}{4} x^2 (1-x)^2 \right) dx =$$

$$= \left(-\tfrac{1}{40}(1-x)^5 + \tfrac{1}{12}(1-x)^4 - \tfrac{1}{12}(1-x)^3 - \tfrac{1}{4}\left(\tfrac{1}{3}x^3 - \tfrac{1}{2}x^4 + \tfrac{1}{5}x^5 \right) \right) \Big|_0^1 =$$

$$= 0 - \tfrac{1}{4}\left(\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} \right) - \left(-\tfrac{1}{40} + \tfrac{1}{12} - \tfrac{1}{12} \right) = \tfrac{1}{40} + \tfrac{1}{120} = \tfrac{1}{30}.$$

Ex. Find an expression for the mass of the solid bounded by the sphere $x^2 + y^2 + z^2 = 1$ whose density at each point is proportional to the distance from the origin

Solution: Since the density is proportional to the distance from the origin, it has the form $f(x,y,z) = k\sqrt{x^2+y^2+z^2}$ for some constant k. The sphere is x-, y-, and z- simple. Treating it as x-simple, it can be described by

$$-\sqrt{1-y^2-z^2} \leq x \leq \sqrt{1-y^2-z^2}, \quad (y,z) \in B$$

where B is the unit disk in the yz-plane. But B is given by the inequalities

$$-\sqrt{1-z^2} \leq y \leq \sqrt{1-z^2}, \qquad -1 \leq z \leq 1$$

so the mass is given by the triple integral

$$\iiint_S f = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} k\sqrt{x^2+y^2+z^2}\, dx\, dy\, dz$$

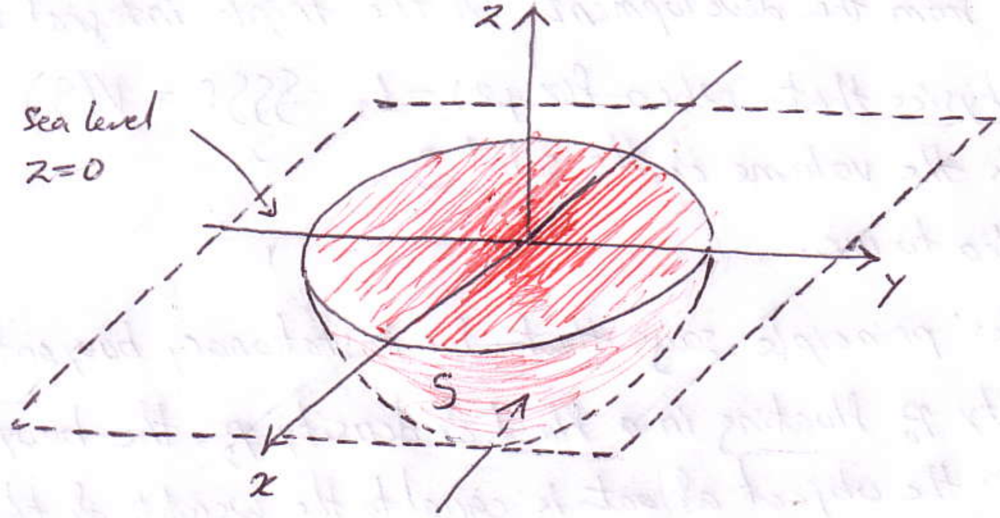
We could evaluate this integral by brute force using integral tables and a calculator, but we will instead wait until we learn how to transform coordinates and evaluate the integral with much greater ease.

Ex. (Gilligan's Island) Find an expression for the mass of blood with spill radius R, $M(R)$.

Solution: The density of blood at a given point (x,y,z) is equivalent to the product of blood density and its concentration:

$$1060\, C_R(x,y,z) = 1060\, \frac{\left(\sqrt{x^2+y^2+z^2} - R\right)^{16}}{R^{16}}, \text{ where}$$

(x,y,z) belongs to the lower half of the sphere $x^2+y^2+z^2 = R^2$



$$x^2+y^2+z^2=R^2, \quad z\le 0$$

Notice that the lower half of the sphere, S is z-simple and described by

$$-\sqrt{R^2-x^2-y^2} \le z \le 0, \quad (x,y)\in B$$

where B is the disc in the xy plane with center at the origin and radius R. This disc can be described by the inequalities

$$-\sqrt{R^2-x^2} \le y \le \sqrt{R^2-x^2}$$

$$-R \le x \le R.$$

So the mass of blood, $M(R)$, is given by the triple integral

$$\iiint_S \rho = \int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{-\sqrt{R^2-x^2-y^2}}^{0} 1060\,\frac{\left(\sqrt{x^2+y^2+z^2}-R\right)^{16}}{R^{16}}\,dz\,dy\,dx$$

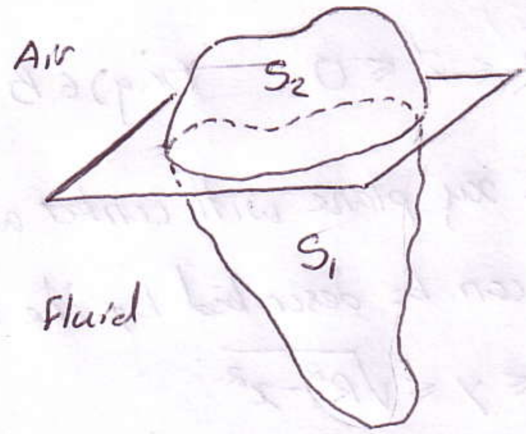
Again, we wait until we learn about the change of variables theorem to make evaluation of the integral significantly simpler.

It is evident from the development of the triple integral from a concept in physics that when $f(x,y,z) = 1$, $\iiint_S f = V(S)$, where $V(S)$ is the volume of the solid S.

We put this idea to use.

Ex. Archimedes' principle says that for a stationary buoyant object of density ρ_0 floating in a fluid of density ρ_f, the buoyant force that keeps the object afloat is equal to the weight of the fluid displaced. Use this to show that the fraction of the solid above the fluid level is $\frac{\rho_f - \rho_0}{\rho_f}$.

Solution:



Let S be the solid region in space occupied by the object, S_1 the portion submerged, and S_2 the portion above the fluid level. Since the buoyant force is the weight of the solid, $\iiint_S \rho_0 g$, and since the weight of the displaced fluid is $\iiint_{S_1} \rho_f g$, by Archimedes' principle,

$$\iiint_S \rho_0 g = \iiint_{S_1} \rho_f g.$$

Since the densities are constant, we have

$$\rho_0 g \iiint_S 1 = \rho_f g \iiint_{S_1} 1 \quad \text{and} \quad \iiint_{S_1} 1 = \frac{\rho_0}{\rho_f} \iiint_S 1$$

Now the fraction of the object above the fluid level is

$$\frac{\iiint_{S_2} 1}{\iiint_S 1} = \frac{\iiint_S 1 - \iiint_{S_1} 1}{\iiint_S 1} = \frac{\iiint_S 1 - \rho_0 \rho_f \iiint_S 1}{\iiint_S 1} =$$

$$= 1 - \frac{\rho_0}{\rho_f} = \frac{\rho_f - \rho_0}{\rho_f}.$$

Triple integrals possess all the properties that we listed for double integrals. These are summarized in the following theorem.

Thm: Let f and g be integrable functions, and c be a constant.

1. $\iiint_S (f+g) = \iiint_S f + \iiint_S g$
2. $\iiint_S cf = c \iiint_S f$
3. $\iiint_{S_1 \cup S_2} f = \iiint_{S_1} f + \iiint_{S_2} f$, where $S_1 \cap S_2 = \emptyset$.
4. (Mean Value Theorem for Triple Integrals) If f is cont. on a solid region S with volume $V(S)$, then there exists a point $(x_0, y_0, z_0) \in S$ such that

$$\iiint_S f = f(x_0, y_0, z_0)\, V(S)$$

5. If $f(x,y,z) \le g(x,y,z)$ for $(x,y,z) \in S$, then

$$\iiint_S f \le \iiint_S g.$$

6. If f is cont. and satisfies $f(x,y,z) > 0$ for all $(x,y,z) \in S$, where S has nonempty interior, then

$$\iiint_S f > 0$$

Ex. Without calculating the integral, explain why

$$\iiint_S (x+y+z) = 0$$

where S is the solid between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$

Solution: The integrand $f(x,y,z) = x+y+z$ is an "odd" function in that $f(-x,-y,-z) = -x-y-z = -(x+y+z) = -f(x,y,z)$. If we slice S into halves S_1 and S_2 with a plane through the origin, then for each point (x,y,z) in S_2 there corresponds exactly one point $(-x,-y,-z)$ in S_1. From this it follows, by the oddness of the integrand, f, and property (2) of the theorem above that

$$\iiint_{S_2} f(x,y,z) = \iiint_{S_1} f(-x,-y,-z) = \iiint_{S_1} -f(x,y,z) =$$

$$= -\iiint_{S_1} f(x,y,z)$$

By property (3) of the same theorem

$$\iiint_S (x+y+z) = \iiint_{S_1} (x+y+z) + \iiint_{S_2} (x+y+z) =$$

$$= \iiint_{S_1} (x+y+z) - \iiint_{S_1} (x+y+z) = 0$$